ECON 5110 Class Notes
Learning

1 Introduction

This section relies heavily on the material in George Evans and Seppo Honkapohja's book *Learning and Expectations in Macroeconomics.* Expectations of future economic variables play an important role in macroeconomic theory. Examples include the permanent-income lifecycle consumption hypothesis, monetary policy and asset pricing models. The evolution of expectations in macroeconomics can be classified as follows:

- Naive expectations. Under this mechanism, expectations of a future variable y_{t+1} are given by $y_{t+1}^e = y_t$.

- Adaptive expectations. An example of adaptive expectations (AE) is

$$y_{t+1}^e = y_t^e + \lambda(y_t - y_t^e)$$

  where the parameter λ governs how current expectations adjust to the previous period's forecasting errors. AE were commonly used in Keynesian models that dominated the macro landscape in the 1960s and 1970s. For example, the expectations-augmented Phillips curve often employed AE. In terms of policy, AE imply that policymakers can continually adjust policy instruments (such as government spending or the money supply) to manipulate macro aggregates.

- Rational expectations. The rational expectations (RE) revolution in macroeconomics began in the mid 1970s with the research of Robert Lucas and Thomas Sargent. It has dominated macroeconomic theory ever since. RE assumes economic agents are very sophisticated. They form expectations of future variables according to

$$y_{t+1}^e = E(y_{t+1}|\Omega_t)$$

  where E is the mathematical expectation operator and Ω_t is the information set containing all information dated at time t and earlier. RE assumes agents know the structure of the economy and all relevant parameter values. In terms of policy, RE imply that policymakers are no longer able systematically manipulate macro aggregates – agents understand policymakers' incentives to do so and adjust their behavior accordingly.

- Learning. Learning in macroeconomics is a reaction to the strong assumptions made with RE. In particular, it seems unreasonable to assume that economic agents know the relevant parameter values with certainty when even the best econometricians must themselves estimate the parameters. Learning generally assumes that, while agents are able to figure out the reduced-form equations governing the economy, they must continually update their estimates of the parameters. An interesting question is whether, through the learning process, agents can grope their way toward the rational expectations equilibrium (REE). Learning is also a useful tool to choose between multiple REE – only those that are stable under learning would be expected to be observed.

2 Learning Techniques

2.1 The Setup

Begin by considering a structural macroeconomic model (similar to the one discussed in the previous set of lecture notes):

$$\begin{aligned} y_t &= a + b_1 E^*_{t-1} y_t + b_2 E^*_{t-1} y_{t+1} + c x_t \\ x_t &= \rho x_{t-1} + e_t \end{aligned} \tag{1}$$

where E^*_{t-1} is some arbitrary expectations mechanism and $E^*_{t-1} e_t = 0$. The REE solution takes the form

$$y_t = \bar{\phi}_0 + \bar{\phi}_1 x_{t-1} + \bar{\eta}_t \tag{2}$$

where $\bar{\eta}_t = c e_t$. Assume now that agents do not know $(\bar{\phi}_0, \bar{\phi}_1)$, but are able to figure the structure of equation (2). Agents instead specify a perceived law of motion (PLM)

$$y_t = \phi_0 + \phi_1 x_{t-1} + \eta_t \tag{3}$$

where (ϕ_0, ϕ_1) are the agents estimates of $(\bar{\phi}_0, \bar{\phi}_1)$. The actual law of motion (ALM) is found by substituting the forecasts for y_t and y_{t+1} from the PLM into the structural model (1):

$$y_t = a + b_1[\phi_0 + \phi_1 x_{t-1}] + b_2[\phi_0 + \phi_1 \rho x_{t-1}] + c x_t,$$

which after rearranging gives

$$y_t = [a + \phi_0(b_1 + b_2)] + [\phi_1(b_1 + \rho b_2) + c\rho]x_{t-1} + \bar{\eta}_t. \tag{4}$$

This implicitly defines a mapping from the PLM to the ALM

$$T\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} = \begin{pmatrix}a + \phi_0(b_1 + b_2)\\ \phi_1(b_1 + \rho b_2) + c\rho\end{pmatrix}$$

The relevant questions are whether the parameters in equation (4) converge under reasonable learning rules (discussed below), and if so, do they converge to the REE $(\bar{\phi}_0, \bar{\phi}_1)$?

2.2 Least-Squares Learning

Let $\phi_{t-1} = (\phi_{0,t-1}, \phi_{1,t-1})$ be the estimate of $\phi = (\phi_0, \phi_1)$ at time $t-1$, and let $z_i = (1, x_i)'$. The least-squares estimates are

$$\begin{pmatrix}\phi_{0,t-1}\\ \phi_{1,t-1}\end{pmatrix} = \left[\sum_{i=1}^{t-1} z_{i-1}z_{i-1}'\right]^{-1}\left[\sum_{i=1}^{t-1} z_{i-1}y_i\right]. \tag{5}$$

Note that the least squares estimates can also be written in a recursive manner as

$$\begin{aligned}\phi_t &= \phi_{t-1} + t^{-1}R_t^{-1}z_{t-1}(y_t - \phi_{t-1}'z_{t-1}) \\ R_t &= R_{t-1} + t^{-1}(z_{t-1}z_{t-1}' - R_{t-1}),\end{aligned} \tag{6}$$

which is known as recursive least squares (RLS). Here t^{-1} is referred to as the gain, $(y_t - \phi_{t-1}'z_{t-1})$ is the most recent forecast error and R_t is the moment matrix for z_t. Substituting these estimates back into (4), we have

$$y_t = [a + \phi_{0,t-1}(b_1 + b_2)] + [\phi_{1,t-1}(b_1 + \rho b_2) + c\rho]x_{t-1} + \eta_t$$

or alternatively

$$y_t = T(\phi_{t-1})'z_{t-1} + \eta_t. \tag{7}$$

Substituting (7) into (6) gives

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1}((T(\phi_{t-1})' - \phi'_{t-1}) z_{t-1} + \eta_t) \tag{8}$$

$$R_t = R_{t-1} + t^{-1}(z_{t-1} z'_{t-1} - R_{t-1}), \tag{9}$$

which is a recursive stochastic system. Showing convergence of this recursive least squares system is complicated (see chapter 6 of Evans and Honkapohja) and by no means obvious. Under learning, economic variables depend on agents' econometric forecasts of a system, which in turn depends on their forecasts. This type of learning environment can lead to either divergence from or convergence to REE. Fortunately, the concept of expectational stability (E-stability) can be used to establish convergence (or lack thereof).

2.3 Expectational Stability

Before presenting the conditions necessary for E-stability, first note that the REE solution $(\bar{\phi}_0, \bar{\phi}_1)$ is a fixed point of the mapping $\phi = T(\phi)$. We will show this explicitly in an example below. We say the REE is E-stable if the REE is locally asymptotically stable under the differential equation

$$\frac{d}{d\tau}\begin{pmatrix}\phi_0 \\ \phi_1\end{pmatrix} = T\begin{pmatrix}\phi_0 \\ \phi_1\end{pmatrix} - \begin{pmatrix}\phi_0 \\ \phi_1\end{pmatrix} \tag{10}$$

where τ denotes artificial time. In other words, an REE is E-stable if small deviations from an REE under a perceived law of motion and a given learning rule, gradually return back to the REE. Using the framework above, we would look for the conditions under which the equations

$$\begin{aligned}\frac{d\phi_0}{d\tau} &= a + \phi_0(b_1 + b_2) - \phi_0 = a + \phi_0(b_1 + b_2 - 1) \\ \frac{d\phi_1}{d\tau} &= \phi_1(b_1 + \rho b_2) + c\rho - \phi_1 = \phi_1(b_1 + \rho b_2 - 1) + c\rho\end{aligned}$$

generate stability in a neighborhood of the REE. Assuming that $0 \leq \rho \leq 1$, a sufficient condition for E-stability is $b_1 + b_2 < 1$.

3 Economic Applications

3.1 Cobweb Model

Consider a competitive market for a single good. The demand for the good is given by

$$d_t = \alpha_0 - \alpha_1 p_t + \nu_t^d$$

and since there is a production lag, supply depends on expected price

$$s_t = \beta_0 + \beta_1 E_{t-1}^* p_t + \nu_t^s$$

where ν_t^d and ν_t^s are mutually uncorrelated, mean-zero white-noise shocks. Assuming markets clear (i.e., $d_t = s_t$), then we have the reduced-form equation

$$p_t = a + bE_{t-1}^* p_t + \eta_t$$

where $a = (\alpha_0 - \beta_0)/\alpha_1$, $b = -\beta_1/\alpha_1 < 0$, and η_t is mean-zero white noise.

3.1.1 Naive Expectations

Under naive expectations ($E_{t-1}^* p_t = p_{t-1}$), we have

$$p_t = a + bp_{t-1} + \eta_t. \tag{11}$$

There are two cases:

1. <u>Irregular case</u>. If the supply curve is steeper than the demand curve (i.e., $0 > b > -1$), then equation (11) is a stationary stochastic process, the equilibrium is indeterminate and the fixed-point $p = (1-b)^{-1}a$ is a "sink". (Note: This appears to be inconsistent with the results in the previous set of notes, but notice that equation (11) is written in its backward-looking, as opposed to forward-looking, form).

2. <u>Regular case</u>. Conversely, if the demand curve is steeper than the supply curve (i.e., $b < -1$), the unique, fundamental equilibrium is $p_t = (1-b)^{-1}a + \tilde{\eta}_t$, a noisy steady-state.

3.1.2 Rational Expectations

Under rational expectations ($E^*_{t-1}p_t = E_{t-1}p_t$), we have

$$p_t = a + bE_{t-1}p_t + \eta_t. \tag{12}$$

Begin by taking expectations of both sides of (12) conditional on information at time $t-1$, which gives $E_{t-1}p_t = (1-b)^{-1}a$. Substituting back into (12), produces the unique REE

$$\begin{aligned} p_t &= (1-b)^{-1}a + \eta_t \\ &= \bar{\phi} + \eta_t. \end{aligned} \tag{13}$$

3.1.3 Stability of the REE Under LS Learning

Assume that agents do not know $\bar{\phi}$ in equation (13), but instead use the following PLM to forecast prices:

$$p_t = \phi + \eta_t.$$

Plugging the forecasts back into equation (12), gives the ALM

$$p_t = (a + b\phi) + \eta_t.$$

The condition for E-stability (and hence convergence under least-squares learning) is that $\bar{\phi}$ be locally, asymptotically stable under

$$\frac{d\phi}{d\tau} = T(\phi) - \phi = (a + b\phi) - \phi = a + \phi(b-1). \tag{14}$$

Therefore, if $b < 1$ the REE is stable under learning. Since $b < 0$, the REE is indeed stable under learning. For example, in the unlikely event that supply sloped down more steeply than demand (i.e., $\beta_1 < 0$ and $|\beta_1| > |\alpha_1|$), then the REE solution would be unstable under learning. Finally, note that $\bar{\phi} = (1-b)^{-1}a$ can easily be seen as the unique fixed point of (14).

3.2 Lucas Aggregate Supply Model

Lucas' aggregate supply function is

$$q_t = q + \pi(p_t - E^*_{t-1}p_t) + \epsilon_t \tag{15}$$

where q_t is aggregate output, p_t is the price level, $\pi, q > 0$ and ϵ_t is mean-zero white noise. Aggregate demand is derived from the quantity equation

$$m_t + v_t = p_t + q_t \tag{16}$$

where v_t is a velocity shock and m_t is the money supply, which is white noise around a constant mean m

$$m_t = m + \mu_t. \tag{17}$$

All variables are measured in logarithms. Some simple algebra produces the reduced form

$$p_t = a + bE^*_{t-1}p_t + \eta_t \tag{18}$$

where

$$a = \frac{m-q}{1+\pi},\ b = \frac{\pi}{1+\pi} \text{ and } \eta_t = \frac{1}{1+\pi}(\mu_t + \nu_t - \epsilon_t).$$

Since equation (18) is in the same form as the Cobweb equation, it has the same condition for stability under learning

$$b < 1 \Longrightarrow \pi < (1+\pi).$$

This condition is satisfied so that the REE from the Lucas supply model is always stable under learning.

3.3 Ramsey Model

3.3.1 Framework

Consider a discrete-time version of the Ramsey growth model, which abstracts from population growth, technology shocks and depreciation. Labor supply (N_t) is normalized to one. The representative agent maximizes

$$E^*_t \textstyle\sum_{i=0}^{\infty} \beta^{t+i}(1-\sigma)^{-1}C^{1-\sigma}_{t+i}$$

subject to

$$C_t + K_{t+1} = w_t + (1 + r_t)K_t.$$

Firms, given the CRS production function $f(K_t) = K_t^\alpha$, maximize profits given by

$$f(K_t) - r_t K_t - w_t.$$

This produces the standard Euler equations

$$\begin{aligned} r_t &= f'(K_t) \\ w_t &= f(K_t) - K_t f'(K_t). \end{aligned}$$

Plugging these into the consumer's problem (and assuming perfect foresight) gives

$$\begin{aligned} C_{t+1} &= C_t[\beta(1 + \alpha(K_t + K_t^\alpha - C_t)^{\alpha-1})]^{1/\sigma} \\ K_{t+1} &= K_t + K_t^\alpha - C_t. \end{aligned}$$

The Ramsey model has a unique equilibrium, involving a saddle path that converges to a non-stochastic steady state $(\bar{C}, \bar{K})$. In other words, for a given K_0, there is a unique choice of C_0 that will put the economy on a convergent path to the steady state. All other choices for C_0 will lead to divergent paths that violate some non-negativity constraint or transversality condition.

3.3.2 Learning

Now let's introduce some uncertainty and learning. Given the knife-edge nature of the equilibrium, it is an open question as to whether the economy will converge to the rational expectations equilibrium when agents start with non-rational expectations and use some sort of adaptive learning. Begin by linearizing the system

$$\hat{c}_t = a_1 E_t^* \hat{c}_{t+1} + a_2 E_t^* \hat{k}_{t+1} \quad (19)$$

$$\hat{k}_{t+1} = b_1 \hat{c}_t + b_2 \hat{k}_t. \quad (20)$$

The REE takes the form

$$\begin{aligned}\hat{k}_{t+1} &= \bar{\phi}_1 \hat{k}_t \\ \hat{c}_{t+1} &= \bar{\phi}_2 \hat{k}_{t+1}.\end{aligned}$$

Now assume that agents use the following PLM in forecasting

$$\begin{aligned}\hat{k}_{t+1} &= \phi_1 \hat{k}_t \\ \hat{c}_{t+1} &= \phi_2 \hat{k}_{t+1}.\end{aligned}$$

Substituting agents' forecasts of $\hat{c}_{t+1}$ and $\hat{k}_{t+1}$ into (19) gives the ALM

$$\begin{aligned}\hat{c}_{t+1} &= [\phi_1(a_2 + a_1\phi_2)]\hat{k}_{t+1} \\ \hat{k}_{t+1} &= [b_1\phi_2 + b_2]\hat{k}_t.\end{aligned}$$

The ALM can also be written as

$$\begin{aligned}\hat{c}_t/\hat{k}_t &= [\phi_{1,t}(a_2 + a_1\phi_{2,t})] = T_1(\phi_{1,t}, \phi_{2,t}) \\ \hat{k}_{t+1}/\hat{k}_t &= [b_1\phi_{2,t} + b_2] = T_2(\phi_{1,t}, \phi_{2,t}).\end{aligned}$$

The law of motion for the parameters under adaptive learning is

$$\begin{aligned}\phi_{1,t} &= \phi_{1,t-1} + \gamma_t[(\hat{c}_{t-1}/\hat{k}_{t-1}) - \phi_{1,t-1}] = \phi_{1,t-1} + \gamma_t[T_1 - \phi_{1,t-1}] \\ \phi_{2,t} &= \phi_{2,t-1} + \gamma_t[(\hat{k}_t/\hat{k}_{t-1}) - \phi_{2,t-1}] = \phi_{2,t-1} + \gamma_t[T_2 - \phi_{2,t-1}].\end{aligned}$$

Convergence to the REE can be verified computationally. Evans and Honkapohja state that the convergence to $(\bar{\phi}_1, \bar{\phi}_2)$ is rapid for parameter values $\beta = 0.9$, $\alpha = 0.3$, $\sigma = 0.5$ and constant-gain learning. Because $|\bar{\phi}_1| < 1$, this also implies that (C_t, K_t) converge to $(\bar{C}, \bar{K})$. So, the Ramsey REE appears to be stable under learning.