The Delta Method

Daniel Ratajczak Tristin Waggener

Department of Economics
University of Wyoming

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Brief Overview

The Delta Method states that we can approximate the asymptotic behavior of functions over a random variable, if the random variable is itself asymptotically normal. In practice, this theorem tells us that even if we do not know the expected value and variance of the function $g(X)$ we can still approximate it reasonably.
The delta method gives us the asymptotic distribution of an asymptotically normal random variable.
The delta method is designed for random variables that converge to a normal distribution, and then you can take functions of that and it still converges to a normal distribution.

Review of Taylor Series

The Taylor Series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots ,$$

where $n!$ denotes the factorial of n. Using sigma notation, this can be written as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ denotes the *nth* derivative of f evaluated at the point a.

Example: The Taylor Series of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

In sigma notation, this can be written as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example: The Taylor Series of e^x

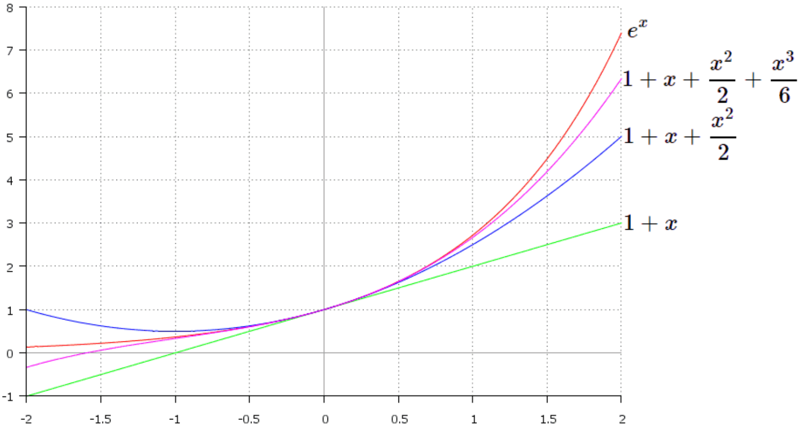

Figure: 3 Taylor Approximations of the function $g(x) = e^x$.

Review of Central Limit Theorem

Let $X_1, \cdots, X_n$ denote a random sample from any distribution with finite mean μ and finite variance σ^2. Then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0, 1)$$

Review of Slutsky's Theorem

If X_n and Y_n are two sequences of random variables such that $X_n \xrightarrow{p} c$ and $Y_n \xrightarrow{d} Y$ then:

$$
\begin{aligned}
&1. \quad X_n + Y_n \xrightarrow{d} c + Y. \\
&2. \quad X_n Y_n \xrightarrow{d} cY. \\
&3. \quad \frac{Y_n}{X_n} \xrightarrow{d} \frac{Y}{c} \quad \text{for } c \neq 0.
\end{aligned}
$$

$Y_n \xrightarrow{d} Y$, then for any continuous function $g(y)$ that does not depend on n, $g(Y_n) \xrightarrow{d} g(Y)$.

Delta Method Theorem

Let X_n be a sequence of random variables such that $\sqrt{n}(X_n - \theta) \xrightarrow{d} N(0, \sigma^2) \equiv X$ where θ, σ^2 are finite constants, then

$$\sqrt{n}(f(X_n) - f(\theta)) \xrightarrow{d} N(0, \sigma^2(f'(\theta))^2)$$

where $f'(\theta)$ exists and is non-zero.[2]

Delta Method Procedure

Suppose that we have a sequence of random variables X_n, such that as $n \to \infty$

$$X_n \sim N(\mu, \frac{\sigma^2}{n})$$

We can rearrange this statement as follows:

$$(X_n - \mu) \sim \frac{\sigma}{\sqrt{n}} N(0, 1)$$
$$\frac{\sqrt{n}(X_n - \mu)}{\sigma} \sim N(0, 1)$$

Delta Method Procedure (Continued)

If g is some smooth function, then the Delta Method states that:

$$\frac{\sqrt{n}(g(X_n) - g(\mu))}{|g'(\mu)|\sigma} \approx N(0,1)$$

Rearranging, we can write:

$$g(X_n) \approx N\left(g(\mu), \frac{g'(\mu)^2\sigma^2}{n}\right)$$

The above statement is an approximation because

$$g(X_n) = g(\mu) + g'(\mu)(X_n - \mu) + g''(\mu)\frac{(X_n - \mu)^2}{2!} + \cdots$$

is an infinite sum. The Delta Method ignores higher order terms.

Proof of Delta Method

Given Taylor's Theorem[3], we know that so long as g is continuous and differentiable up to the *kth* derivative, where $k \geq 2$, then at the point μ:

$$g(X_n) \approx g(\mu) + g'(\mu)(X_n - \mu)$$

Subtracting $g(\mu)$ from both sides, we obtain:

$$(g(X_n) - g(\mu)) \approx g'(\mu)(X_n - \mu)$$

Proof of Delta Method(Continued)

We know by the central limit theorem and our assumptions regarding X_n that $(X_n - \mu) \xrightarrow{d} N(0, \frac{\sigma^2}{n})$. Therefore, we can write:

$$(g(X_n) - g(\mu)) \approx g'(\mu)N(0, \frac{\sigma^2}{n})$$

Finally, we can use the properties of the normal distribution (multiplying by a constant, adding a constant) to write:

$$g(X_n) \approx N\left(g(\mu), \frac{g'(\mu)^2\sigma^2}{n}\right)$$

Delta Method: Multidimensional Case

The Delta Method also generalizes to multidimensional functions, where instead of converging on the standard normal the random variable must converge in distribution to a multivariate normal, and the derivatives of g are replaced with the gradient of g (a vector of all partial derivatives).

$$\nabla g = \begin{bmatrix} \frac{dg}{dx_1} \\ \frac{dg}{dx_2} \\ \vdots \\ \frac{dg}{dx_n} \end{bmatrix}$$

Example of Delta Method using the Binomial Distribution

$$X_i \sim Bin(1, p)$$
$$E(X_i) = p$$
$$Var(X_i) = p(1-p)$$
$$f(x) = x(1-x)$$
$$f'(x) = 1 - 2x$$

Applying the delta method we can now write:

$$\frac{\sqrt{n}(\overline{X} - p)}{\sqrt{p(1-p)}} \sim N(0,1)$$
$$\frac{\sqrt{n}(f(\overline{X}) - f(p))}{\sqrt{p(1-p)} \cdot f'(p)} \sim N(0,1)$$

We can find the probability as follows:

$$p(\overline{X}(1-\overline{X}) \leq t) \equiv \Phi\left(\frac{\sqrt{n}(t - p(1-p))}{p(1-p) \cdot |1-2p|}\right)$$

MATLAB Simulation

The MPC (Marginal Propensity to Consume) is estimated using consumption and income data.
We can use the following formula:

$$Consumption = Autonomous + MPC * Income$$

The government spending multiplier (gsm) is calculated as:

$$gsm = \frac{1}{1 - MPC}$$

MATLAB Simulation (Continued)

To get standard errors using the Delta Method, first find the gradient vector:

$$g = \begin{bmatrix} 0 \\ \frac{1}{(1-MPC)^2} \end{bmatrix}$$

which is a vertical $kx1$ vector. The "mean" or estimate of the multiplier is just the formula $\frac{1}{(1-MPC)}[1]$. The variance is:

$$g'[s^2(X'X)^{-1}]g$$

,which collapses to a $1x1$ point estimate.

MATLAB Simulation (Continued)

We can write the variance of the multiplier as:

$$var(multiplier) = \begin{bmatrix} 0 \\ \frac{1}{(1-MPC)^2} \end{bmatrix}' \left[s^2(X'X)^{-1}\right] \begin{bmatrix} 0 \\ \frac{1}{(1-MPC)^2} \end{bmatrix}$$

,where the middle term is the variance-covariance matrix from OLS.
Because this transformation only uses one parameter, the variance can also be calculated without matrix form as:

$$g = \frac{1}{(1-MPC)^2}$$

$$var(multiplier) = g^2 var(MPC)$$

Plot of Keynesian Multiplier Sampling Distribution with Superimposed Delta Method PDF

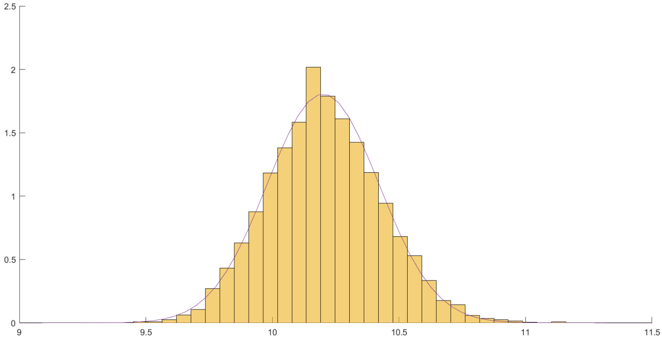

Figure: Keynesian Multiplier Sampling Distribution with Superimposed Delta Method PDF.

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