The Delta Method

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Outline

- Motivation
  - A Generalized Central Limit Theorem
  - Taylor's Series Approximation
- The Delta Method
  - Assumptions
  - Univariate Case
  - Multivariate Case
- Numerical Study in MATLAB

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A Generalized Central Limit Theorem

Application of the Central Limit Theorem (CLT) gives a way of evaluating the limiting distribution of $\bar{X}_n$:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$

Now let us consider instead, the limiting distribution of a linear transformation of $\bar{X}_n$, like $g(\bar{X}_n) = a\bar{X}_n + b$. By the linearity of the expectation operator we know that

$$E[g(\bar{X}_n)] = a\mu + b = g(\mu).$$

If we apply the same stabilizing transformation we know that

$$\sqrt{n}[g(\bar{X}_n) - g(\mu)] = \sqrt{n}[(a\bar{X}_n) + b - (a\mu + b)] = a\sqrt{n}(\bar{X}_n - \mu).$$

Thus, the limiting distribution of $\sqrt{n}[g(\bar{X}_n) - g(\mu)]$ is $N(0, a^2\sigma^2)$.

A Generalized Central Limit Theorem

- A limiting distribution of a linear transformation of $\bar{X}_n$ is straight forward, but what if $g(t)$ is nonlinear?
- The simple algebra shown above alone is no longer enough. However, it does lay the foundation for the further generalization into nonlinear functions.
- Using the fact $\bar{X}_n$ is consistent for μ, we know that $\bar{X}_n$ will be very close to μ for a large n.
- Thus, the only parts of the nonlinear transformation that are meaningful are those in a small neighborhood around μ.

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Taylor's Series Approximation

A First Order Taylor's Series Approximation of g gives us a linear approximation of the the behavior of g in the small neighborhood around μ:

$$g(t) \approx g(\mu) + g'(\mu)(t - \mu).$$

We can now apply the same logic used in the case of the linear transformation, giving us

$$\sqrt{n}[g(\bar{X}_n) - g(\mu)] \xrightarrow{d} N(0, \sigma^2(g'(\mu))^2).$$

The expression above is a special case of the Delta Method.

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Assumptions

- In general the Delta Method is used for evaluating the asymptotic behavior of a nonlinear transformation of a random variable.
- A special case of the Delta Method is what is most often used in econometrics.
- This special case relies on three assumptions:
  - plim(θ_n) = θ.
  - $\sqrt{n}(\theta_n - \theta) \xrightarrow{d} N(0, \sigma^2)$.
  - $g(\theta_n)$ is a continuous and continuously differentiable function that does not involve n.

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Univariate Case

Consider that an estimator θ_n and function $g(x)$ that adhere to the above assumptions i.e.

$$\sqrt{n}(\theta_n - \theta) \xrightarrow{d} N(0, \sigma^2).$$

Then the univariate delta method gives us

$$\sqrt{n}(g(\theta_n) - g(\theta)) \xrightarrow{d} N(0, \sigma^2(g'(\theta))^2).$$

Univariate Example

For the nonlinear transformation $g(x) = log(x)$ and i.i.d. observations $\{x_i\}$ from a sample of size n we know

$$\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$

The delta method gives the limiting distribution

$$\sqrt{n}(log(\bar{x}_n) - log(\mu)) \xrightarrow{d} N\left(0, \left(\frac{\sigma}{\mu}\right)^2\right).$$

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Multivariate Case

If $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k$ are k consistent estimators of k parameters $\theta_1, \theta_2, ..., \theta_k$ with asymptotic covariance matrix

$$\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1k} \\ v_{21} & v_{22} & \dots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \dots & v_{kk} \end{bmatrix},$$

and if $f(\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k)$ is a continuous function with continuous derivatives then the delta method gives us the asymptotic variance of $f(\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k)$:

$$\mathbf{g'Vg} = \begin{bmatrix} \frac{\partial f(.)}{\partial \theta_1} & \frac{\partial f(.)}{\partial \theta_2} & \dots & \frac{\partial f(.)}{\partial \theta_k} \end{bmatrix} \mathbf{V} \begin{bmatrix} \frac{\partial f(.)}{\partial \theta_1} \\ \frac{\partial f(.)}{\partial \theta_2} \\ \dots \\ \frac{\partial f(.)}{\partial \theta_k} \end{bmatrix}.$$

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MATLAB: Keynesian Consumption Function

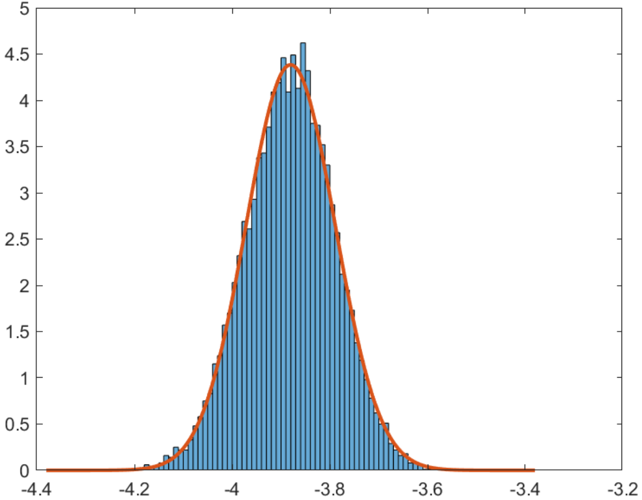
$$C = \beta_1 + \beta_2(Y - T) + \epsilon$$

C = Personal Consumption Expenditure (1960-2018)
Y = Gross National Income (1960-2018)
T = Federal Government Tax Receipts (1960-2018)

$$\text{Tax Multiplier} = \frac{-MPC}{1 - MPC} = \frac{-\beta_2}{1 - \beta_2}$$

MATLAB: Results

Figure: Histogram of the Monte Carlo Tax Multiplier Estimates (Sample Size of 59)



MATLAB: Results

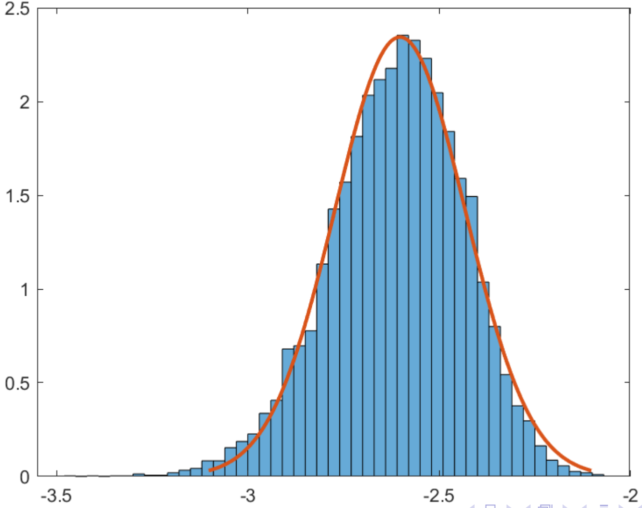


Figure: Histogram of the Monte Carlo Tax Multiplier Estimates (Sample Size of 20)

Sources

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