Recursive Residuals

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    - Serial correlation
    - Heteroskedasticity

Methodology

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- Record residual, add the latest observation to the basis set, and repeat
- The OLS estimate that is used to estimate the jth observation is

$$\hat{\beta}_{j-1} = (X'_{j-1}X_{j-1})^{-1}X'_{j-1}Y_{j-1}$$

Formula for the Recursive Residuals

The recursive residual w_j for the observation y_j is defined as

$$w_j = (y_j - x_j'\hat{\beta}_{j-1}).$$

Standardized Recursive Residuals

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$$w_j^* = \frac{(y_j - x_j'\hat{\beta}_{j-1})}{(1 + x_j'(X_{j-1}'X_{j-1})^{-1}x_j)^{1/2}}.$$

Recursive Residuals & Structural Change

Recursive Residuals & Structural Change

- Constant β vs changing β

Recursive Residuals & Structural Change

- Constant β vs changing β
- Identifying structural change
    - CUSUM
    - CUSUMSQ
   - MOSUM
    - MOSUMSQ

CUSUMSQ

The Cumulative Sum of Squares method calculates the values

$$WW_r = \frac{\sum_{j=p+1}^{r} w_j^{*2}}{\sum_{j=p+1}^{n} w_j^{*2}}, \quad r = p+1, \cdots n.$$

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- $WW_r \sim \beta(r-p, n-r)$
- $E[WW_r] = \dfrac{r-p}{n-p}$

Distribution of WW_r
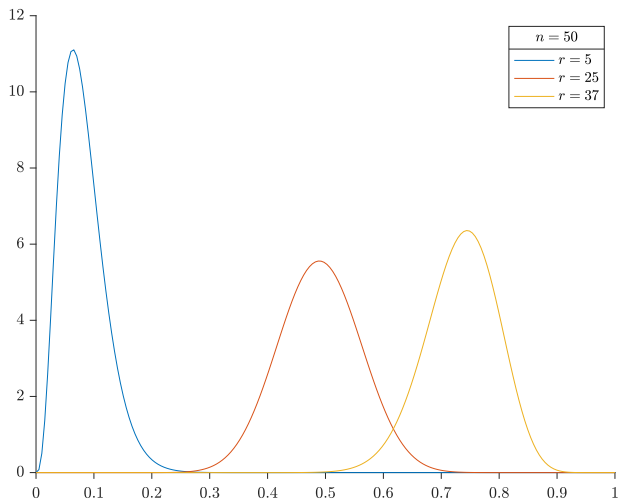
Identifying Structural Change with CUSUMSQ

- H_0 : β is constant across X
- H_a : β is not constant across X
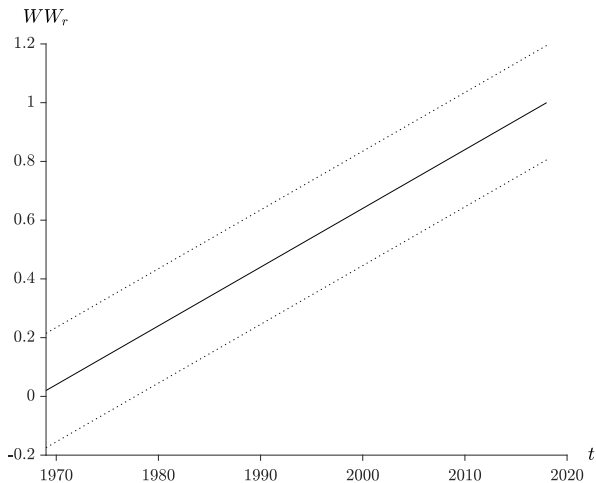
MATLAB Example
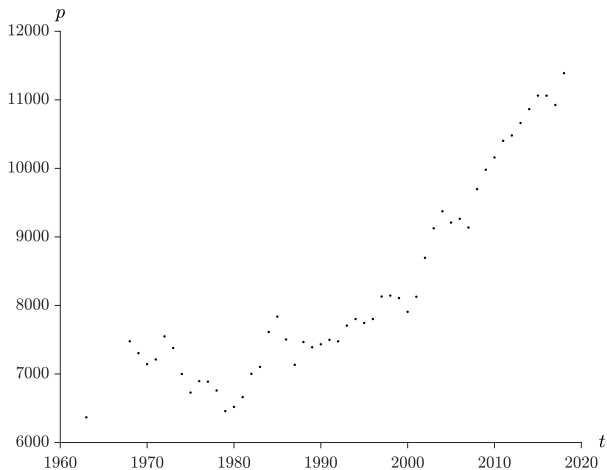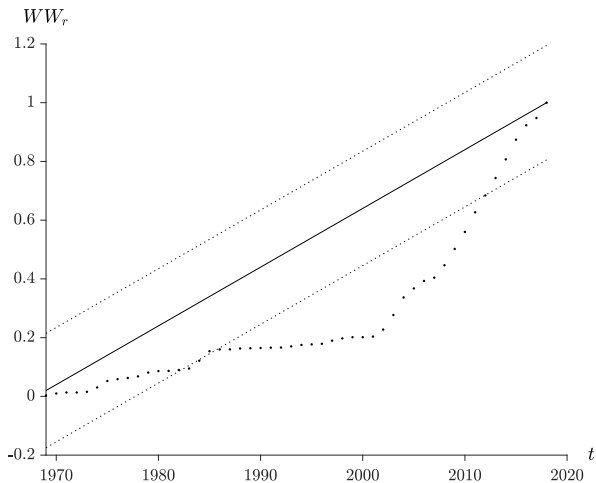

Figure: College tuition over time
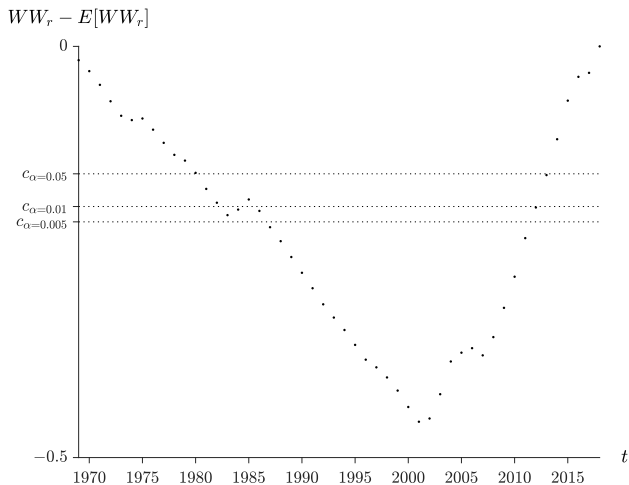
Plot of $E[WW_r]$...
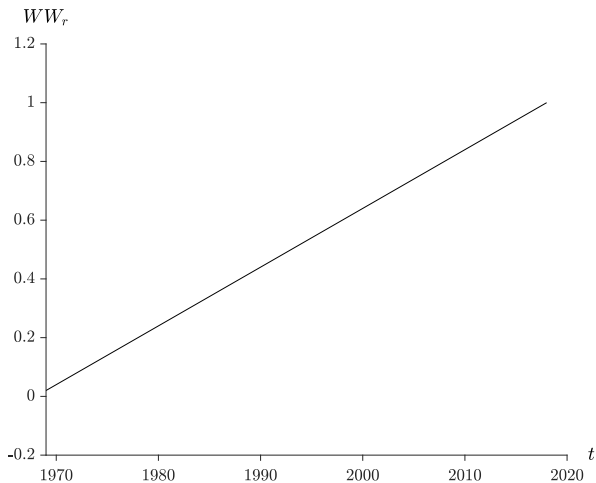
...with Critical Regions

Plot of Cumulative Sum of Squares

Selecting Kink Points

Validation

Run OLS with kink points and test for statistical significance of "spline" model vs OLS.

Questions?

Citations

Kianifard, Farid, and William H. Swallow. "A review of the development and application of recursive residuals in linear models." *Journal of the American Statistical Association* 91, no. 433 (1996): 391-400.

Harvey, Andrew C. *The econometric analysis of time series*. Mit Press, 1990.