ECON 5350 Solutions to the Final Exam – Fall 2012

Consider the Keynesian consumption function

$$C_t = \alpha + \beta Y_t + \epsilon_t$$

where C_t is aggregate consumption, Y_t is disposable income, and $t = 1, ..., T$.

1. (20 pts) Under what condition is the OLS estimate of β unbiased? consistent? efficient? Also, describe the sampling distribution for the OLS estimator of β.

   Solution.

   - The OLS estimate is unbiased if X is fixed in repeated sampling and $E(\epsilon_t) = 0$.
   - The OLS estimate is consistent if $\lim_{t\to\infty} X'X = Q$ is a finite positive-definite matrix, $\lim_{t\to\infty} E(X'\epsilon) = 0$, and $\lim_{t\to\infty} var(X'\epsilon) = 0$ where X is the $T \times 2$ independent variable matrix.
   - The OLS estimate is efficient if it is unbiased and $var(\epsilon) = \sigma^2 I$.
   - Provided the errors are normally distributed, the sampling distribution for the OLS estimate of β is also normally distributed with mean β and variance $\sigma^2 / \sum_{t=1}^{T}(Y_t - \bar{Y})^2$.

2. (20 pts) Describe three methods for testing the null hypothesis that $\alpha = \beta$. Which is best?

   Solution. Three methods are the unrestricted F test, restricted F test, and a t test. They are all equivalent.

3. (20 pts) Describe all the required steps to directly test the hypothesis that the government spending multiplier is equal to two. Describe an equivalent method of indirectly testing the same hypothesis. Which method is preferable and why?

   Solution. The government spending multiplier is $\delta = 1/(1 - \beta)$. The test statistic for the direct test is

$$W = \frac{(\hat{\delta} - 2)^2}{var(\hat{\delta})},$$

   which has an asymptotic chi-square distribution with one degree of freedom. The linearized estimate of $\hat{\delta}$ is

$$\hat{\delta} \simeq \delta + (1 - \beta)^{-2}(b - \beta)$$

   and the variance is

$$var(\hat{\delta}) = (1 - \beta)^{-4} var(b).$$

   An indirect method would test that $\beta = 0.5$ using a t or F test. The latter method is better because it does not involve any approximation error.

4. (20 pts) Consider a permanent structural break in the variance of the errors at $t = t_*$. Describe the procedure for obtaining an efficient estimate of β. How does this procedure compare to separate OLS estimation before and after $t = t_*$? Explain.

Solution. The efficient estimator is GLS

$$\hat{\beta}_{2\times 1} = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)$$

where X and Y are the independent and dependent variable matrices. The partitioned variance-covariance matrix of the errors is

$$\Omega_{T\times T} = \begin{bmatrix} \sigma_1^2 I_{t_*} & 0 \\ 0 & \sigma_2^2 I_{T-t_*} \end{bmatrix}.$$

The estimator for the slope can also be written as

$$\hat{\beta} = \frac{\sum_{t=1}^{t_*}(C_t - \bar{C})(Y_t - \bar{Y})/\sigma_1^2 + \sum_{t=t_*+1}^{T}(C_t - \bar{C})(Y_t - \bar{Y})/\sigma_2^2}{\sum_{t=1}^{t_*}(Y_t - \bar{Y})^2/\sigma_1^2 + \sum_{t=t_*+1}^{T}(Y_t - \bar{Y})^2/\sigma_2^2}.$$

OLS estimation before and after $t = t_*$ will provide two estimates of the slope. Each will be unbiased and efficient for the subsample. However, the GLS estimator is preferable because it is also efficient, consistent, and based on a larger sample.

5. (20 pts) Now consider an alternative model:

$$C_t = \ln[\alpha + \beta Y_t] + \epsilon_t.$$

Outline a procedure for least squares estimation of β using the Gauss-Newton and Newton-Raphson algorithms.

Solution. The Gauss-Newton procedure starts by linearizing the above regression model

$$C_t = \ln[\alpha_0 + \beta_0 Y_t] + \frac{1}{\alpha_0 + \beta_0 Y_t}(\alpha - \alpha_0) + \frac{Y_t}{\alpha_0 + \beta_0 Y_t}(\beta - \beta_0) + \epsilon_t^*$$

re-arranging

$$C_t - \ln[\alpha_0 + \beta_0 Y_t] + \frac{\alpha_0}{\alpha_0 + \beta_0 Y_t} + \frac{\beta_0 Y_t}{\alpha_0 + \beta_0 Y_t} = \frac{1}{\alpha_0 + \beta_0 Y_t}\alpha + \frac{Y_t}{\alpha_0 + \beta_0 Y_t}\beta + \epsilon_t^*.$$

and iteratively estimating with OLS until convergence. The NLS estimate with Newton-Raphson minimizes

$$S(\alpha, \beta) = \sum_{t=1}^{T}(C_t - \ln[\alpha + \beta Y_t])^2$$

using the algorithm

$$\hat{\theta}_{s+1} = \hat{\theta}_s - H(\hat{\theta}_s)^{-1} g(\hat{\theta}_s)$$

to iterate until convergence, where $\theta = (\alpha, \beta)'$. The gradient and Hessian are given by

$$\begin{aligned}
g(\hat{\theta}_s) &= \begin{bmatrix} \frac{\partial S(\alpha,\beta)}{\partial \alpha} = -2\sum_t (C_t - \ln[\alpha + \beta Y_t])^2 / [\alpha + \beta Y_t] \\ \frac{\partial S(\alpha,\beta)}{\partial \beta} = -2\sum_t Y_t (C_t - \ln[\alpha + \beta Y_t])^2 / [\alpha + \beta Y_t] \end{bmatrix} \text{ and} \\
H(\hat{\theta}_s) &= \begin{bmatrix} \frac{\partial^2 S(\alpha,\beta)}{\partial \alpha^2} & \frac{\partial^2 S(\alpha,\beta)}{\partial \alpha \partial \beta} \\ \frac{\partial^2 S(\alpha,\beta)}{\partial \alpha \partial \beta} & \frac{\partial^2 S(\alpha,\beta)}{\partial \beta^2} \end{bmatrix}.
\end{aligned}$$