ECON 5350 Class Notes
Generalized Regression Model

1 Introduction

In this section, we extend the classical multiple linear regression model to allow for heteroscedasticity and autocorrelation. This is sometimes referred to as the generalized regression model. The generalized regression model can be written as

$$Y = X\beta + \epsilon \tag{1}$$

where $E(\epsilon) = 0$ and $E(\epsilon\epsilon') = \sigma^2\Omega$. For example, the symmetric $(n \times n)$ variance-covariance matrix for ϵ may look like

$$\begin{bmatrix} \sigma_1^2 & \rho_1 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \rho_1 & \sigma_2^2 & \cdots & \rho_{n-3} & \rho_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{n-2} & \rho_{n-3} & \cdots & \sigma_{n-1}^2 & \rho_1 \\ \rho_{n-1} & \rho_{n-2} & \cdots & \rho_1 & \sigma_n^2 \end{bmatrix}$$

where σ_i^2 represents heteroscedasticity and ρ_i represents autocorrelation.

2 Properties of OLS Estimators

2.1 Review of Classical Linear Regression Model Results

Assuming that $E(\epsilon) = 0$ and $E(\epsilon\epsilon') = \sigma^2 I$, we have learned that the OLS estimator b is

- unbiased,
- consistent,
- efficient and
- asymptotically normal.

If we add that $\epsilon \sim N$, then b is also

- asymptotically efficient.

2.2 Finite-Sample Properties of OLS Estimators

Let's find the mean and variance of the OLS estimator, b, under the generalized regression model.

- $b = (X'X)^{-1}(X'Y) = (X'X)^{-1}(X'(X\beta + \epsilon)) = \beta + (X'X)^{-1}X'\epsilon$. Therefore, $E(b) = \beta$.
- $var(b) = E[(b-\beta)(b-\beta)'] = E[(X'X)^{-1}X'\epsilon\epsilon'X(X'X)^{-1}] = (X'X)^{-1}X'\sigma^2\Omega X(X'X)^{-1}$.

Therefore, if $\epsilon \sim N$, we know that

$$b \sim N[\beta, (X'X)^{-1}X'\sigma^2\Omega X(X'X)^{-1}].$$

2.3 Asymptotic Properties of OLS Estimators

Let's begin by examining the consistency of b under the generalized regression model. Using the principle of convergence in mean square, we know that

$$\lim_{n\to\infty} E(b) = \beta$$

and

$$\lim_{n\to\infty} var(b) = \lim_{n\to\infty} (\frac{1}{n}X'X)^{-1}(\frac{\sigma^2}{n})(\frac{1}{n}X'\Omega X)(\frac{1}{n}X'X)^{-1} = 0$$

so long as $\text{plim}\frac{1}{n}(X'X) = Q$ and $\text{plim}\frac{1}{n}(X'\Omega X) = \tilde{Q}$, both finite positive definite matrices. Therefore, $\text{plim}(b) = \beta$ and the OLS estimator is consistent. Furthermore, application of the CLT and assuming X and Ω satisfy certain regularity conditions (see Greene, pp. 260), we know that

$$b \overset{asy}{\sim} N[\beta, \frac{\sigma^2}{n}Q^{-1}\tilde{Q}Q^{-1}].$$

It is instructive to note that the standard results from the Classical linear regression model are recovered if we let $\Omega = I$, so that the Classical linear regression model is a special case of the generalized model.

3 Efficient Estimation When Ω is Known

3.1 Generalized Least Squares (GLS)

The goal is to find an efficient estimator (smallest variance in the unbiased class) of β given Ω. Begin by noting that because Ω is symmetric, we can diagonalize the matrix by pre-multiplying by P and post-multiplying by P', that is

$$P\Omega P' = I$$

where $P = \Lambda^{-1/2}C'$, C is a matrix of characteristic vectors and Λ is a diagonal matrix with all the eigenvalues of Ω along the main diagonal. Therefore, using simple matrix algebra, we know that $\Omega^{-1} = P'P$.

Now multiply (1) by P

$$PY = PX\beta + P\epsilon$$

and redefine so that $Y_* = PY$, $X_* = PX$ and $\epsilon_* = P\epsilon$. We can rewrite the regression model as

$$Y_* = X_*\beta + \epsilon_* \tag{2}$$

where $\epsilon_* = P\epsilon \sim iid(0, P\sigma^2\Omega P' = \sigma^2 I)$. Therefore, (2) satisfies all the Classical assumptions and

$$\begin{aligned}\hat{\beta} &= (X_*'X_*)^{-1}(X_*'Y_*) \\ &= (X'P'PX)^{-1}(X'P'PY) \\ &= (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)\end{aligned}$$

is BLUE. Here are a few notes about the GLS estimator:

1. $\hat{\beta}$ is a consistent estimator so long as $\text{plim}(\frac{1}{n}X_*'X_*) = Q_*$, a finite positive definite matrix.
2. $\hat{\beta} \overset{asy}{\sim} N[\beta, \sigma^2(X_*'X_*)^{-1} = \sigma^2(X'\Omega^{-1}X)^{-1}]$.
3. The problem of minimizing $\epsilon_*'\epsilon_* = \epsilon'\Omega^{-1}\epsilon$ can be interpreted as <u>weighted least squares</u>.
4. R^2 from GLS is not directly comparable to the OLS case.
5. $s^2 = e'e/(n-k)$ is a biased estimator of σ^2 but $s_*^2 = e_*'e_*/(n-k)$ is an unbiased estimator.

3.2 Consequences of Using Ordinary Least Squares

Assume that $\epsilon \sim N(0, \sigma^2\Omega)$ and $\Omega \neq I$. The consequences of using OLS include

1. b and $\hat{\beta}$ are both unbiased but $var(b) = \sigma^2(X'X)^{-1}(X'\Omega X)(X'X)^{-1} > \sigma^2(X'\Omega^{-1}X)^{-1} = var(\hat{\beta})$.
2. $s^2 = e'e/(n-k)$ is a biased estimator of σ^2.
3. Standard hypothesis testing is invalid (i.e., the t and F statistics do not have t and F distributions).

3.3 Consumption Function – A Monte Carlo Experiment

In this example, we are going to simulate artificial consumption data from the following simple model

$$c_t = \beta_1 + \beta_2 y_t + \epsilon_t$$

where $t = 1, ..., T$. The variance-covariance matrix for ϵ is

$$\frac{\sigma^2}{1-\rho^2}\begin{bmatrix} \sigma_1^2/\sigma^2 & \rho & \cdots & \rho^{T-2} & \rho^{T-1} \\ \rho & \sigma_2^2/\sigma^2 & \cdots & \rho^{T-3} & \rho^{T-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{T-2} & \rho^{T-3} & \cdots & \sigma_{T-1}^2/\sigma^2 & \rho \\ \rho^{T-1} & \rho^{T-2} & \cdots & \rho & \sigma_T^2/\sigma^2 \end{bmatrix}.$$

The population values are $\beta_1 = 100$ and an MPC equal to $\beta_2 = 0.8$. See MATLAB example 15 for further details.

4 Estimation When Ω is Unknown

Typically, the econometrician will not know Ω, and therefore it needs to be estimated along with the other parameters.

4.1 Feasible GLS Estimation

The GLS estimator was given by $\hat{\beta} = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)$. The feasible GLS estimator is given by $\hat{\beta}_{FGLS} = (X'\hat{\Omega}^{-1}X)^{-1}(X'\hat{\Omega}^{-1}Y)$. In general, the finite-sample properties of $\hat{\beta}_{FGLS}$ are unknown. However, if the elements of Ω are all consistently (not necessarily efficiently) estimated, then $\hat{\beta}_{FGLS}$ can be shown to be asymptotically efficient. Note that in small samples, it is possible that b will outperform $\hat{\beta}_{FGLS}$ when $\Omega \neq I$.

4.2 Hypothesis Testing

An asymptotically valid hypothesis test for H_0: $R\beta = q$ is

$$J \cdot F = \frac{(R\hat{\beta}_{FGLS} - q)'[R(X'\Omega^{-1}X)^{-1}R']^{-1}(R\hat{\beta}_{FGLS} - q)}{\hat{\sigma}^2_{FGLS}} \overset{asy}{\sim} \chi^2(J).$$