Delta Method

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Taylor Series Expansion

Theorem: If $h^r(a) = \frac{d^r h(x)}{dx^r}$ when $x = a$ exists, then

$$\lim_{x \longrightarrow a} \frac{h(x) - T_r(x)}{(x-a)^r} = 0$$

The first-order Taylor series approximation of h about θ is

$$h(t) = h(\theta) + \sum_{i=1}^{k} h_i'(\theta)(t_i - \theta_i) + \text{Remainder}$$

Example: Approximate Mean and Variance

Suppose X is a random variable with $E[x] = \mu \neq 0$. If we want to estimate a function $h(\mu)$, a first-order Taylor approximation would give us

$$h(X) = h(\mu) + h'(\mu)(X - \mu).$$

Thus, if we use h(X) as an estimator of $h(\mu)$, we can say that approximately

$$E[h(X)] \approx h(\mu)$$
$$Var[h(X)] \approx h'(\mu)^2 Var(X)$$

Why do we do this?

Suppose, we take $h(\mu) = \frac{1}{\mu}$ with μ unknown. If we estimate $\frac{1}{\mu}$ with $\frac{1}{X}$, we can say

$$E\left(\frac{1}{X}\right) \approx \frac{1}{\mu},$$
$$Var\left(\frac{1}{X}\right) \approx \left(\frac{1}{\mu}\right)^4 Var(X)$$

We can generalise this into a convergence result just like the Central Limit Theorem (CLT). This result is known as the Delta Method. [1]

[1] The variance formula uses the fact that the derivative of $\frac{1}{\mu}$ is $-\frac{1}{\mu^2}$ which is then squared.

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Theorem: Let Y_n be a sequence of random variables that satisfies $\sqrt{n}(Y_n - \theta) \to \mathcal{N}(0, \sigma^2)$ in distribution. For a given function and a specific value of θ, suppose $h'(\theta)$ exists and is not 0. Then,

$$\sqrt{n}[h(Y_n) - h(\theta)] \to \mathcal{N}(0, \sigma^2 h'(\theta)^2)$$

Matlab Simulation

In matlab, we will now simulate the delta method at work. To do this we use an arbitrary regression model

$$Y = x_1 + \beta_2^2 x_2 + \beta_3 x_3 + \epsilon.$$

With a delta function of $\hat{\delta} = b_2^2$ and the corresponding Taylor series

$$\hat{\delta} \approx \delta^0 + \underbrace{h(b_2^0)}_{(2b_2^0)}(b - b_2^0)^1.$$

This means that the variance of a random set of data that we need to compare to the variance of the delta function is

$$\text{var}(\hat{\delta}) = \underbrace{[h(b_2^0)]^2}_{(4(b_2^0)^2)} \underbrace{\text{var}(b)}_{(\sigma^2(x'x)^{-1}_{(2,2)})}$$

Statistical Approximation

We now turn our first-order Taylor series expansion into a statistical approximation such that $t = T$. This gives

$$h(T) = h(\theta) + \sum_{i=1}^{k} h_i'(\theta)(T_i - \theta_i) + \text{Remainder} \tag{1}$$

Taking expectations on both sides, we get

$$\begin{aligned} E[h(T)] &\approx E[h(\theta) + \sum_{i=1}^{k} h_i'(\theta)(T_i - \theta_i)] \\ &\approx h(\theta) + \sum_{i=1}^{k} h_i'(\theta)E[T_i - \theta_i] \end{aligned} \tag{2}$$

Statistical Approximation

We can also approximate our variance as

$$\begin{aligned} Var[h(T)] &\approx E[(h(T) - h(\theta))^2] && \text{(from(1))} \\ &\approx E[(\sum_{i=1}^{k} h_i'(\theta)(T_i - \theta_i))^2] && \text{(from(2))} \\ &= \sum_{i=1}^{k} h_i'(\theta)^2 Var(T_i) + 2\sum_{i>j} h_i'(\theta)h_j'(\theta)Cov(T_i, T_j) && (3) \end{aligned}$$

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Proof

The Taylor series expansion around $Y_n = \theta$ is

$$h(Y_n) = h(\theta) + h'(\theta)(Y_n - \theta) + \text{Remainder},$$

where the remainder $\to 0$ as $Y_n \to \theta$. From the assumption that Y_n satisfies the CLT, we have $Y_n \to \theta$ in probability, so it follows that the remainder $\to 0$ as well. Rearranging our Taylor Series expansion around $Y_n = \theta$, we have

$$\sqrt{n}(h(Y_n) - h(\theta)) = h'(\theta)\sqrt{n}(Y_n - \theta) + \text{Remainder}.$$

Applying Slutsky's Theorem, with $W_n = h'(\theta)\sqrt{n}(Y_n - \theta)$ and Z_n as the remainder, we have the RHS converging to $\mathcal{N}(0, \sigma^2 h'(\theta)^2)$.