Structural Change and Recursive Residuals

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Breaks in Model Parameter Stability

- We begin with a model estimate of relationships between variables:

$$y_t = \beta_0 + \beta_1 x_t + \mu_t$$

- A break in the data can skew the model.
- This problem occurs frequently in time series data.
- Result = Poor forecasting

Testing Parameter Stability

- Is the break point or period known?
- If it is known, then the Chow test can be used.
- F-test using sub-periods and the entire sample.
- Dummy variables can also be used around the sub-periods.

Chow Test

$$F - Test_{Chow} = \frac{RSS_{rest} - RSS_{unrest}}{RSS_{unrest}} * \frac{T - 2k}{k}$$

- T = number of observations.
- k = number of parameters being tested.
- Unrestricted RSS is from the original model, including the structural break.
- Less change = more stability.

When the break point is unknown

- Chow test works well when break point is readily identifiable.
- Often can only identify a range, but not know exact month, quarter, period.
- Can repeat the Chow test multiple times with different break points - largest F-stat wins (Quandt Likelihood Ratio Test - QLR).
- Or, perform Recursive Residuals Estimation.

Recursive Residuals

- Recursive Least Squares can be used to identify a *suspected* structural break.
- Begin with small sample of the full model.
- Sequentially add one period at a time, and run regression on the new, expanded, sample.
- Repeated until entire data set is included.
- Results are then plotted, between stats and graph the break should become apparent, if there is one that is statistically significant.
- The rth scaled residual is:

$$w_r = \frac{e_r}{\sqrt{1 + x_r' \left(X_{r-1}' X_{r-1} \right)^{-1} x_r}}.$$

Testing Recursive Estimation - *CUSUM* and *CUSUM*^{2}

- *CUSUM* and *CUSUM*^{2} based on Recursive Least Squares.
- General tests of stability on time series data.
- *CUSUM*^{2} would range from zero at beginning of period to one at the end of the period under "perfect" parameter stability.
- Confidence bands selected to determine whether to reject null hypothesis.

Formulas for *CUSUM* Test

$$W_t = \sum_{r=K+1}^{r+t} \frac{w_r}{\hat{\sigma}},$$

where

$$\hat{\sigma}^2 = \frac{\sum_{r=K+1}^{T} (w_r - \bar{w})^2}{T - K - 1}$$

and

$$\bar{w} = \frac{\sum_{r=K+1}^{T} w_r}{T - K}.$$

Performing the *CUSUM* Test

Plotting W_t against t and confidence bounds given by the two lines connecting

$$[K, \pm a\,(T-K)^{1/2}]$$

and

$$[T, \pm 3a\,(T-K)^{1/2}],$$

we use confidence intervals for a.

- The null hypothesis (that W_t has a mean zero and a variance approximately the number of residuals being summed) is rejected if the plot strays outside the boundaries.

Performing the $CUSUM^2$ Test

Using

$$S_t = \frac{\sum_{r=K+1}^{r=t} w_r^2}{\sum_{r=K+1}^{r=T} w_r^2}$$

the expected value of S_t is approximately $(t-K)(T-K)$.

- The residuals are independent and the test is conducted by constructing confidence bounds for $E[S_t]$ at values of t and plotting the bounds and S_t against t.
- The bounds are found using $E[S] \pm c_0$, with c_0 being dependent on $(T-K)$ and the significance level selected.
- If the cumulative sum strays outside the bounds selected then the hypothesis of parameter stability is rejected.