Delta Method

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It's all about the variance!

- Let b and c denote constants and let X denote a random variable. We want to find the variance of the linear function $(c + bX)$.

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$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
Var(bX) &= E(b^2X^2) - [E(bX)]^2 \\
&= b^2E(X^2) - b^2[E(X)]^2 \\
&= b^2[E(X^2) - E(X)^2] \\
&= b^2 Var(X)
\end{aligned}
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- Let b and c denote constants and let X denote a random variable. We want to find the variance of the linear function $(c + bX)$.
- What is the variance $Var(c)$?
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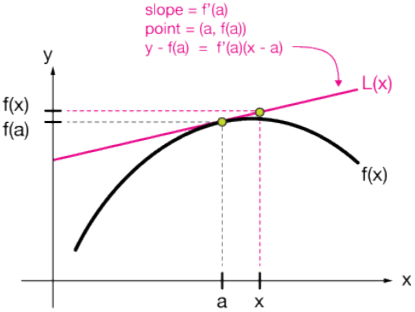
- What is the variance of $Var(bX)$?
- $b^2 Var(X)$
- $$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ Var(bX) &= E(b^2X^2) - [E(bX)]^2 \\ &= b^2E(X^2) - b^2[E(X)]^2 \\ &= b^2[E(X^2) - E(X)^2] \\ &= b^2 Var(X) \end{aligned}$$
- Now, how can we find the variance of cX^2?

Well, that's where we use the Delta Method

What is the Delta Method?

- First, linearize using Taylor Series
- Taylor Series
- Next, find the variance

Do you remember Taylor Series?



- $f(x) \approx f(a) + f'(a)(x - a)$
- Delta Method ignores higher order Taylor Series expansions

How to find variance $f(X) = X^2$?

- linearize it with first-order Taylor approximation
- $f(x) \approx f(a) + f'(a)(x - a)$

$$f(X) \approx a^2 + 2a(X - a)$$
$$Var(a^2 + 2a(X - a))$$

- In general, $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

$$= \underbrace{Var(a^2)}_{=0} + Var(2a(X - a)) + \underbrace{2Cov(aX)}_{=0}$$
$$= Var(2a(X - a))$$
$$= (2a)^2 Var(X)$$

Why do we care?

- Compute standard errors of econometric estimators
- Inference
- Hypothesis testing and t-statistics
- $Y = b_1 + b_2 X_2 + b_3 X_3 + \hat{\epsilon}$
- $b_2 \times b_3$
- Approximate variance using the Delta Method.

Bernoulli Trials - coin flip

- Suppose X is a random variable

$$X = 1 \text{ when the outcome is success}$$
$$X = 0 \text{ when the outcome is a failure}$$

- Let X denote the outcome of getting heads on a coin flip and p be the probability of getting heads on a coin flip.

$$Var(X) = E(X^2) - [E(X)]^2$$
$$E(X) = (p \times 1) + (0 \times (1 - p)) = p$$
$$E(X) = (p \times 1^2) + (0^2 \times (1 - p)) = p$$
$$Var(X) = \underbrace{E(X^2)}_{=p} - \underbrace{[E(X)]^2}_{p^2}$$
$$Var(X) = p - p^2 = p(1 - p)$$

Estimating the odds

- Sometimes we are interested in calculating the odds of something given by $\frac{p}{1-p}$
- How do we calculate $Var(\frac{\hat{p}}{1-\hat{p}})$?
- First, we linearize using Taylor Series expansion
- In general, $f(x) \approx f(a) + f'(a)(x-a)$

$$Var(\frac{\hat{p}}{1-\hat{p}}) \approx Var(\frac{\hat{p}_o}{1-\hat{p}_o}) + Var((\frac{1}{1-\hat{p}_o})^2 \times (\hat{p} - p_o))$$

$$Var(\frac{\hat{p}}{1-\hat{p}}) \approx \underbrace{Var(\frac{\hat{p}_o}{1-\hat{p}_o})}_{=0} + (\frac{1}{1-\hat{p}_o})^2 \times Var(\hat{p})$$

$$Var(\frac{\hat{p}}{1-\hat{p}}) = (\frac{1}{1-\hat{p}})^2 \times \hat{p}(1-\hat{p})$$

$$Var(\frac{\hat{p}}{1-\hat{p}}) = \frac{\hat{p}}{(1-\hat{p})^3}$$

Simple Keynesian Model

Consumption Function

$$C = a + bY$$

C = Consumption

a = Autonomous Consumption

b = MPC

Y = Disposable Income

Simple Keynesian Model

Consumption Function

$$C = a + bY$$

$$\begin{aligned} C &= \text{Consumption} \\ a &= \text{Autonomous Consumption} \\ b &= \text{MPC} \\ Y &= \text{Disposable Income} \end{aligned}$$

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We can estimate b and find it's SE.

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What about a non-linear transformation of $\hat{b}$?

Non-linear Transformation

The Government Expenditure Multiplier is

$$\frac{dY}{dG} = \frac{1}{1-b}$$

Non-linear Transformation

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Non-linear Transformation

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The sample estimate is

$$\hat{\gamma} = \frac{1}{1-\hat{b}}$$

SE of Government Spending Multiplier

Step 1: Linearize

First-order Taylor approximation gives

$$f(\gamma) \approx f(\gamma_o) + f'(\gamma_o)(\hat{b} - \gamma_o)$$

and γ_o is the point of approximation.

SE of Government Spending Multiplier

Step 1: Linearize

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$$f(\gamma) \approx f(\gamma_o) + f'(\gamma_o)(\hat{b} - \gamma_o)$$

and γ_o is the point of approximation.

$$\Leftrightarrow f(\gamma) \approx \frac{1}{1-\gamma_o} + (1-\gamma_o)^{-2}(\hat{b} - \gamma_o)$$

where $\gamma_o, f(\gamma_o)$ and $f'(\gamma_o)$ are constants.

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SE of Government Spending Multiplier

Step 2: Find Variance

Using Delta Method

$$Var(\hat{\gamma}) = [(1 - \gamma_o)^{-2}]^2 Var(\hat{b})$$

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\begin{aligned}
Var(\hat{\gamma}) &= [(1-\gamma_o)^{-2}]^2 Var(\hat{b}) \\
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Step 2: Find Variance

Using Delta Method

$$\begin{aligned} Var(\hat{\gamma}) &= [(1-\gamma_o)^{-2}]^2 Var(\hat{b}) \\ &= [(1-\gamma_o)]^{-4} Var(\hat{b}) \end{aligned}$$

Thus the standard error is

$$SE(\gamma) = \sqrt{[(1-\gamma_o)]^{-4} Var(\hat{b}))}$$

MATLAB

Data Sources

- Consumption
- Income

Shortcomings

- Not all functions have valid Taylor Series expansions.
- For some, Taylor series diverge from the original functions as the order of the terms increases.

Approximation

- Large sample size

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- As $N- > \infty$, the variance from Delta Method is asymptotically equal to the actual variance.

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- Large sample size
- As $N- > \infty$, the variance from Delta Method is asymptotically equal to the actual variance.
- Good Taylor approximation implies efficient application of the Delta Method.

Alternative

An alternative would be to Bootstrap standard errors

Questions?