RECURSIVE RESIDUALS

So far, we have only seen structural breaks where we know the point at which the break occurs. Now suppose we have a dataset where we don't know if there's a particular policy implemented that would change our regression. In this case:

- Recursive residuals will jump at point of break
- How big of a jump matters? Outside a confidence interval!

Recursive Residual Use. Suppose we have a sample with T observations where the t^{th} recursive residual is the actual prediction error for y_t when the regression is estimated by only $t-1$ observations. Then we have a one step ahead prediction error such that:

$$e_t = y_t - x_t' b_{t-1} \tag{1}$$

- e_t is t^{th} **recursive residual**
- x_t is vector of regressors associated with observation y_t
- b_{t-1} is the least squares coefficient vector computed using $t-1$ observations
- Null hypothesis: mean of e_t is zero

Why Should It Work? By running an OLS regressing on $t-1$ observations, we get an estimate for our b vector.

- If our estimates don't have a structural break:
  - Estimate for e_t using b_{t-1} would be similar to the estimate from e_{t-1}
  - We should continue to see consistent estimators for each e_t
- If structural break:
  - Estimates for e_t should be consistent until point of break
  - At unknown break, error estimate will jump
  - To be considered a structural break, e_t must jump outside confidence interval

If a structural break is found using recursive residuals, we can then implement a tool such as dummy variables or a spline at the time period that the residual jumps out of the confidence interval bound.

Confidence Interval. The confidence interval should be one standard deviation above and below zero. We can find the confidence interval by taking the variance of e_t.

$$\begin{aligned} var(e_t) &= var(y_t - x_t' b_{t-1}) \\ &= var(x_t'\beta + \epsilon_t - x_t' b_{t-1}) \\ &= var(x_t'(\beta - b_{t-1}) + \epsilon_t) \\ &= var(x_t'(\beta - b_{t-1})) + \sigma^2 \end{aligned}$$

$$= \sigma^2 x_t'(X_{t-1}'X_{t-1})^{-1}x_t + \sigma^2$$

Therefore the confidence interval bounds are

$$s.d. = \pm\sqrt{\sigma^2 x_t(X_{t-1}'X_{t-1})^{-1}x_t' + \sigma^2} \tag{2}$$

CUSUM Test. Using recursive residuals, a CUSUM Test is the cumulative sum of residuals.

$$W_t = \sum_{r=K+1}^{r=t} \frac{w_r}{\hat{\sigma}} \tag{3}$$

- w_r is the r^{th} scaled residual
- $\hat{\sigma}$ is the forecast variance of the one step ahead prediction error
- Plot W_t against against t using forecast confidence bounds in the same way we test for recursive residuals
- Null hypothesis: mean of W_t is zero

Matlab Example. See attached data and .m file.