How to Use the Delta Method

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Overview

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Keynesian Consumption

Consider consumption modeled by

$$C = \beta_1 + \beta_2(Y - T) + \epsilon.$$

Goverment spending multiplier

$$\frac{1}{1 - \beta_2}.$$

Tools: Descriptive Statistics

- Mean:
$$\mu = E[X]$$
- Variance:
$$\sigma^2 = Var(X) = E[(X - \mu)^2]$$

Tools: Theorems

Theorem (Central Limit Theorem)

For $X_1, X_2, X_3, ..., X_n$ i.i.d. random variables with finite variance σ^2 we have that

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Theorem (Slutsky's Thoerem)

For sequences of random elements X_n and Y_n,
if X_n converges in distribution to a random element, X,
and Y_n converges in probability to constant, c, then

$$X_n Y_n \xrightarrow{d} cX.$$

Tools: Taylor-Series

First order Taylor series expansion:

$$f(x) = f(x^*) + f'(\tilde{x})(x - x^*)$$

The Delta Method

Theorem (The Delta Method)

For some sequence of random variables b_n,
if $f'(\beta)$ exists and is non-zero valued and
$\sqrt{n}[b_n - \beta] \xrightarrow{d} \mathcal{N}(0, \sigma^2(X'X)^{-1})$,
then,
$\sqrt{n}[f(b_n) - f(\beta)] \xrightarrow{d} \mathcal{N}(0, \sigma^2(X'X)^{-1}(f'(\beta))^2)$.

Proof

Step1:
A first-order Taylor-Series expansion of $f(b_n)$.

$$f(b_n) = f(\beta) + f'(\tilde{\beta})(b_n - \beta)$$

Here $\tilde{\beta}$ lies between b_n and β

Proof Cont.

Step 2:
We know, $b_n \xrightarrow{p} \beta$
Additionally, $b_n < \tilde{\beta} < \beta$.
Hence, $\tilde{\beta} \xrightarrow{p} \beta$.
Therefore, $f'(\tilde{\beta}) \xrightarrow{p} f'(\beta)$.

Proof Cont.

Step 3:
Rearrange:

$$f(b_n) = f(\beta) + f'(\tilde{\beta})(b_n - \beta)$$
$$f(b_n) - f(\beta) = f'(\tilde{\beta})(b_n - \beta)$$
$$\sqrt{n}[f(b_n) - f(\beta)] = f'(\tilde{\beta})\sqrt{n}[b_n - \beta]$$

Proof Cont.

Step 4:
We have,

$$\sqrt{n}[f(b_n) - f(\beta)] = f'(\tilde{\beta})\sqrt{n}[b_n - \beta].$$

We know,

$$\sqrt{n}[b_n - \beta] \xrightarrow{d} \mathcal{N}(0, \sigma^2(X'X)^{-1}).$$

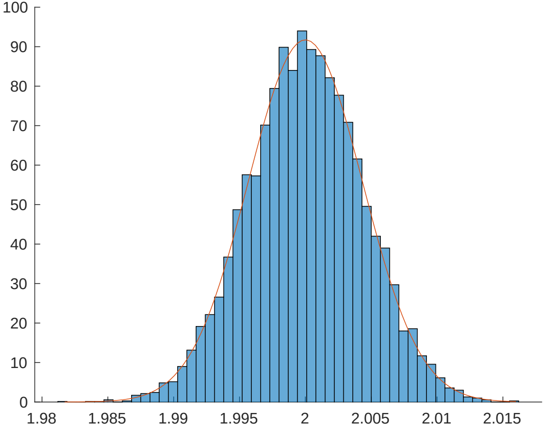
Therefore by Slutskys,

$$\sqrt{n}[f(b) - f(\beta)] \xrightarrow{d} \mathcal{N}(0, \sigma^2(X'X)^{-1}(f'(\beta))^2)$$

Are we sure?

Mean of $f(b_n)$
Variance of $f(b_n)$

Matlab code



The End