ECON 5350 Solution to Midterm Exam – Spring 2007

1. **Generalized Least Squares (25 pts)**. Consider the following partitioned regression model:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \beta + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \tag{1}$$

where

$$\Omega = \begin{bmatrix} \sigma_1^2 I_{n_1} & 0 \\ 0 & \sigma_2^2 I_{n_2} \end{bmatrix}$$

is the variance covariance matrix of the error terms. Find the formula for the GLS estimator and suggest a feasible GLS estimator.

Solution. The GLS estimator is

$$\hat{\beta}_{GLS} = \left(\frac{X_1' X_1}{\sigma_1^2} + \frac{X_2' X_2}{\sigma_2^2} \right)^{-1} \left(\frac{X_1' Y}{\sigma_1^2} + \frac{X_2' Y}{\sigma_2^2} \right)^{-1}.$$

A feasible GLS estimator could be obtained by substituting

$$\hat{\sigma}_j^2 = \frac{(Y_j - X_j b)'(Y_j - X_j b)}{n_j - k}$$

into the GLS estimator, where b is the OLS estimator.

2. **Autocorrelation (30 pts)**. Consider the simple linear regression model $y_t = \beta_0 + \beta_1 x_t + \epsilon_t$, where the errors exhibit second-order autocorrelation: $\epsilon_t = \rho_2 \epsilon_{t-2} + \mu_t$.

(a) Calculate the autocovariance function (i.e., $\gamma(s) = cov(\epsilon_t, \epsilon_{t-s})$) under the assumption that ϵ_t is a weakly covariance stationary process.

Solution. The variance is

$$\gamma(0) = \sigma_\mu^2 / (1 - \rho_2^2).$$

The autocovariance function is

$$\gamma(s) = \rho_2^{s-1} \gamma(0)$$

for $s = \pm 2, \pm 4, \pm 6, ...$ and zero otherwise.

(b) Assuming ρ_2 is known, derive the GLS estimator of β_1.

Solution. The GLS estimator involves estimating the following equation via OLS:

$$(y_t - \rho_2 y_{t-2}) = \beta_0 (1 - \rho_2) + \beta_1 (x_t - \rho_2 x_{t-2}) + \mu_t$$

for $t = 3, 4, ..., T$. The first two observations can either be dropped or scaled in such a way that the variance of the transformed dependent variable equals σ_μ^2. This can be calculated given the information in part (a).

(c) Describe how one would calculate an asymptotically efficient two-step estimator of β_1 if ρ_2 were unknown.

Solution. First we would find a consistent estimate of ρ_2 using an estimator such as

$$\hat{\rho}_2 = \frac{\sum_{t=3}^{T} e_t e_{t-2}}{\sum_{t=1}^{T} e_t^2}.$$

Then we would substitute $\hat{\rho}_2$ into the equation above and estimate via OLS.

3. **SUR and Panel-Data Models (45 pts)**. Consider the following SUR system:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \tag{2}$$

where Y_1, Y_2, ϵ_1 and ϵ_2 are of dimension $(T \times 1)$; X_j is of dimension $(T \times k_j)$ for $j = 1, 2$; β_j is of dimension $(k_j \times 1)$ for $j = 1, 2$; and the system variance-covariance matrix is

$$\Omega = \Sigma \otimes I_T = \begin{bmatrix} \sigma_1^2 I_T & \sigma_{12} I_T \\ \sigma_{12} I_T & \sigma_2^2 I_T \end{bmatrix}$$

where

$$\Omega^{-1} = \Sigma^{-1} \otimes I_T = \begin{bmatrix} a_{11} I_T & a_{12} I_T \\ a_{12} I_T & a_{22} I_T \end{bmatrix}.$$

(a) The SUR system above could be used to incorporate panel data. Describe an estimation strategy to obtain consistent, asymptotically efficient estimates for a fixed effects version of Model (2).

Solution. We could think of each equation as a cross section of length T. If there were n cross sections, then we would need n equations. The first column of each X_j matrix would need to be filled with ones to capture the fixed effects. The first step in obtaining a consistent and asymptotically efficient estimate is to use the OLS residuals to find the following consistent estimate for Σ:

$$\hat{\Sigma} = (1/T) \begin{bmatrix} e_1' e_1 & e_1' e_2 \\ e_1' e_2 & e_2' e_2 \end{bmatrix}.$$

This estimate can then be substituted into the GLS formula

$$\hat{\beta}_{GLS} = [X'\Omega^{-1}X]^{-1}[X'\Omega^{-1}Y].$$

The first element for each $\hat{\beta}_j$ would be the corresponding fixed effect.

(b) Show that equation-by-equation OLS is equivalent to GLS when $X_1 = X_2$. For simplicity, consider the case when $k_1 = k_2 = 1$ and $T = 2$.

Solution. The GLS formula is

$$\begin{aligned}
\hat{\beta}_{GLS} &= [X'\Omega^{-1}X]^{-1}[X'\Omega^{-1}Y] \\
&= \left\{ \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{12} & 0 & a_{22} & 0 \\ 0 & a_{12} & 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix} \right\}^{-1} \times \\
&\quad \left\{ \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{12} & 0 & a_{22} & 0 \\ 0 & a_{12} & 0 & a_{22} \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{bmatrix} \right\} \\
&= \begin{bmatrix} x_1^2 a_{11} + x_2^2 a_{11} & x_1^2 a_{12} + x_2^2 a_{12} \\ x_1^2 a_{12} + x_2^2 a_{12} & x_1^2 a_{22} + x_2^2 a_{22} \end{bmatrix}^{-1} \times \\
&\quad \begin{bmatrix} x_1 y_{11} a_{11} + x_2 y_{12} a_{11} + x_1 y_{21} a_{12} + x_2 y_{22} a_{12} \\ x_1 y_{11} a_{12} + x_2 y_{12} a_{12} + x_1 y_{21} a_{22} + x_2 y_{22} a_{22} \end{bmatrix} \\
&= [1/(x_1^2 + x_2^2)]\Sigma\Sigma^{-1} \begin{bmatrix} x_1 y_{11} + x_2 y_{12} \\ x_1 y_{21} + x_2 y_{22} \end{bmatrix} \\
&= b_{OLS}.
\end{aligned}$$

(c) Describe, in detail, how to test Model (1) versus Model (2) when $\sigma_{12} = 0$. Continue to assume that $X_1 = X_2$, $k_1 = k_2 = 1$ and $T = 2$.

Solution. This can be done as a straightforward F test. If we consider Model (1) as the restricted system and Model (2) as the unrestricted system, then we can estimate both with OLS. Using the restricted (e_*) and unrestricted (e) residuals, we form

$$F = \frac{(e_*'e_* - e'e)/1}{(e'e)/(4-2)},$$

which has an F distribution with one numerator degrees of freedom and two denominator degrees of freedom. The null and alternative hypotheses are

$$H_0 \quad : \quad \beta_1 = \beta_2$$
$$H_A \quad : \quad \beta_1 \neq \beta_2.$$