Distributed Lag Models

Kevin Berry, Sichao Wei

advised by

Professor David Aadland

Department of Economics and Finance
University of Wyoming

April 18, 2013

Examples

In many situations there is an obvious time lag between a decision made by some economic agent and the completion of the action initiated thereby.

1. If a company decides to carry out an investment project, some time will elapse before it is completed.
2. A higher income may cause a family to seek a new apartment but not until the present lease expires.
3. Also, a higher income may cause the household to graduate to a larger size of refrigerator, but if the present one is new, it will probably not be replaced at once.

General Form

Generally, if we only consider one dependent and one explanatory variable, we get a model of the form

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + ... + {e_t}^1, \tag{1}$$

and we assume that the variable x at all times is independent of the stochastic process e_t. In order to keep the discussion simple, we assume hereafter that e_t is a white noise process with zero mean.

[1] We usually use ϵ here, but throughout we will use Judge's notation.

Estimation when the lag length is known (1 of 3)

In this section, the estimation of a model of the general form

$$y_t = \sum_{i=0}^{n^*} \beta_i x_{t-i} + e_t \tag{2}$$

will be discussed. The β_i are the unknown distributed lag coefficients or weights, the x_{t-i} are lagged values of the explanatory variable, $n^*(< \infty)$ is the lag length and e_t is independent white noise. The x_t will be assumed to be nonstochastic for convenience.

Estimation when the lag length is known (2 of 3)

Assuming that T observations on y_t and x_t and n^* **presample values** of x_t are available, the model (2) can be written in matrix notation as

$$Y = X\beta + e, \tag{3}$$

where

$$X = \begin{bmatrix} x_1 & x_0 & \cdots & x_{-n^*+1} \\ x_2 & x_1 & \cdots & x_{-n^*+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_T & x_{T-1} & \cdots & x_{-n^*+T} \end{bmatrix}, \tag{4}$$

$y' = (y_1, ..., y_T)$, $\beta' = (\beta_0, ..., \beta_{n^*})$, and $e' = (e_1, ..., e_T)$.

Estimation when the lag length is known (3 of 3)

1. As a matter of fact, there are $T + n^*$ observations on x_t.
2. $X'X$ is nonsingular for $n^* < T$.
3. If ALL the classical assumptions hold, the OLS estimator $b = (X'X)^{-1}X'Y$ is the best linear unbiased estimator.

Estimation of the lag length (1 of 2)

1. If the lag length n^* is unknown, the number of regressors to be used in the linear model (3) $Y = X\beta + e$ is unknown.
2. $n < n^*$ implies biased estimators for the lag weights (Intuitively, omission of relevant independent variables).
3. $n > n^*$ implies inefficient estimators for the lag weights (Intuitively, inclusion of irrelevant variables, Kennedy:p93).

Estimation of the lag length (2 of 2)

Akaike's (1973) Information Criterion assumes the form

$$AIC(n) = \ln \tilde{\sigma}_n^2 + \frac{2n}{T}, \tag{5}$$

where $\tilde{\sigma}_n^2$ is the maximum likelihood estimator for σ^2 evaluated under the assumption that $n = n^*$, and an estimate $\hat{n}(AIC)$ of n^* is chosen so that AIC assumes its minimum for $n = \hat{n}$. That is, $\hat{n} = \hat{n}_T(AIC)$ is chosen such that

$$AIC(\hat{n}_T(AIC)) = \min\{AIC(n)|n = 0, 1, ..., N_T\}, \tag{6}$$

where N_T is the maximum lag length the investigator is prepared to consider if a sample of size T is available.

The Almon distributed lag (1 of 7)

1. In economic applications, invariably lagged values of an explanatory variable are highly collinear, causing the OLS estimates to have high variances.
2. By far the most popular method employed in this context is the incorporation of extraneous information by specifying a lag distribution.
3. A lag distribution function gives the magnitude of the coefficient of a lagged explanatory variable, expressed as a function of the lag.

The Almon distributed lag (2 of 7)

A popular method of reducing the effect of multicollinearity is proposed by Almon (1965). It is assumed that the lag weights can be represented by a polynomial of degree $q < n^*$,

$$\beta_i = \alpha_0 + \alpha_1 i + ... + \alpha_q i^q, \tag{7}$$

or

$$\beta = H\alpha, \tag{8}$$

where $\alpha' = (\alpha_0, \alpha_1, ..., \alpha_q)$ and

$$H = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n^* & n^{*2} & \cdots & n^{*q} \end{bmatrix}. \tag{9}$$

The Almon distributed lag (3 of 7)

The parameters α may be estimated by substituting (8)$\beta = H\alpha$ into (3)$Y = X\beta + e$ to obtain

$$Y = XH\alpha + e = Z\alpha + e, \tag{10}$$

and the OLS estimator of α is $\hat{\alpha} = (Z'Z)^{-1}Z'Y$. An estimator of β is $\hat{\beta} = H\hat{\alpha}$, if the model is correct. This model has been widely used in applied econometric work because of the flexibility of the polynomial lag shape, the economy in terms of the number of parameters that must be estimated and the ease of estimation.

The Almon distributed lag (4 of 7)

The relation $\beta = H\alpha$ implies that

$$\alpha = H^{+}\beta = (H'H)^{-1}H'\beta, \tag{11}$$

where H^{+} is the **generalized inverse** of H. It follows that

$$\begin{aligned} 0 &= \beta - H\alpha \\ 0 &= \beta - HH^{+}\beta \\ \underbrace{0}_{q} &= \underbrace{(I - H(H'H)^{-1}H')}_{R}\underbrace{\beta}_{\beta} \end{aligned} \tag{12}$$

The estimator $\hat{\beta} = H\hat{\alpha}$, where $\hat{\alpha} = (Z'Z)^{-1}Z'Y$, is equivalent to the restricted least squares estimator obtained by estimating (3) $Y = X\beta + e$ subject to $n^{*} - q$ independent linear homogeneous restrictions of the form $R\beta = 0$ from the $n^{*} + 1$ equations.

Effects of misspecifying the polynomial degree and/or lag length – The Almon distributed lag (5 of 7)

Studies of the consequences of applying incorrect restrictions to a linear model show that

Case	Polynomial Degree	Lag Length	Estimator
1	correct	$n > n^*$	generally biased
2	correct	$n < n^*$	usually biased
3	$q > q^*$	correct	unbiased but inefficient
4	$q < q^*$	correct	always biased

Estimating the polynomial degree when the true lag length is known – The Almon distributed lag (6 of 7)

1. The nested nature of the restrictions associated with increasing the polynomial degree q may be used to construct likelihood ratio tests for the "optimal" polynomial degree.
2. Since a polynomial of degree q may be imposed on the lag weights by specifying $n^* - q$ linear homogeneous restrictions of the form $R\beta = 0$ on the model $Y = X\beta + e$, these restrictions may be viewed as hypotheses.
3. The appropriate test statistic is

$$u = \frac{b'R'[R(X'X)^{-1}R']^{-1}Rb}{(n^* - q)\hat{\sigma}^2} \sim F(n^* - q, T - n^* - 1). \quad (13)$$

Estimating both the lag length and polynomial degree – The Almon distributed lag (7 of 7)

1. One could use a two-step procedure and apply the method introduced above (e.g., Akaike's Information Criterion) to estimate the lag length first and then determine the polynomial degree [Pagano and Hartley (1981)].

Model specificaiton

The general form of infinite distributed lag model is

$$\begin{aligned} y_t &= \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} \ldots + e_t \\ &= (\beta_0 + \beta_1 L + \beta_2 L^2 + \ldots) x_t + e_t \\ &= \beta(L) x_t + e_t \end{aligned} \tag{14}$$

1. The major problem in working with a lag model is the specification of the operator of $\beta(L)$. The most popular infinite lag distribution is the Koyck geometric distributed lag.

2. Roughly there are two approaches to specifying a lag model: (1) a theory-based approach and (2) a data-based approach.

Summery of the infinite distributed lag models

Lag Scheme	Lag Function	Remarks
Geometric	$\beta(L) = \frac{\alpha(1-\lambda)}{1-\lambda L}$	$0 < \lambda < 1$
Pascal	$\beta(L) = \frac{\alpha(1-\lambda)^m}{(1-\lambda L)^m}$	$0 < \lambda < 1$; m=positive integer; $m = 1$,it is geometric;
Discounted Polynomial	$\beta(L) = \frac{\gamma(L)}{(1-\lambda L)^{r+1}}$	$0 \leq \lambda < 1$ $\gamma(L)$ is a polynomial in the lag operator of order r
Rational	$\beta(L) = \frac{\gamma(L)}{\phi(L)}$	$\gamma(L)$ and $\phi(L)$ are polynomials in the lag operator L of order r and m, receptively
Gamma	$\beta(L) = \alpha \sum_{i=0}^{\infty} i^{s-1} \exp(-i) L^i$	
Exponential	$\beta(L) = \alpha \sum_{i=0}^{\infty} \exp\left(\sum_{k=1}^{m} p_k i^k\right) L^i$	$p_m < 0$

Gauss Application

- Gauss Example with a short Monte Carlo Experiment
  - Demonstrate Arithmetic and Almon Polynomial Finite Lags
    - Demonstrate consequences of misspecification
    - Provide procedure for fitting an unknown number of lags and degrees
  - Demonstrate a Geometric Infinite Lag
    - Walk through the Koyck process for estimating an infinite lag

Reference

1. Judge, George G. 1985. *The Theory and Practice of Econometrics*. New York: Wiley.
2. Kennedy, Peter. 2008. *A Guide to Econometrics*. Cambridge, Mass: MIT Press.

Derivation of equation (11)

In equation (11), $H^+ = (H'H)^{-1}H'$, where H^+ is the generalized inverse of H. Note that the dimension of H is $(n+1)\times(q+1)$, and that of H' is $(q+1)\times(n+1)$, such that both $HH'_{(n+1)\times(n+1)}$ and $H'H_{(q+1)\times(q+1)}$ are invertible in the normal sense.

By the definition of generalized inverse,

$$\begin{aligned}
HH^+H &= H \\
H'HH^+HH' &= H'HH' \\
(H'H)H^+(HH') &= H'(HH') \\
(H'H)H^+(HH')(HH')^{-1} &= H'(HH')(HH')^{-1} \\
(H'H)H^+ &= H' \\
(H'H)^{-1}(H'H)H^+ &= (H'H)^{-1}H' \\
H^+ &= (H'H)^{-1}H'
\end{aligned}$$