ECONOMETRICS II - ECON 5360

CENSORED (TOBIT) REGRESSION MODEL

Question: What's the difference between these two graphs? (see whiteboard)

Truncated Random Variable

(Greene, Theorem 19.1) Density of a Truncated Random Variable
If a continuous random variable x has a pdf $f(x)$ and a is a constant, then

$$f(x|x > a) = \frac{f(x)}{Prob(x > a)} \tag{1}$$

Using the fact that

$$Prob(x > a) = 1 - \Phi(\frac{a - \mu}{\sigma}) = 1 - \Phi(\alpha) \tag{2}$$

Where ϕ is the pdf scaled by the appropriate amount and Φ is the CDF.

We can then rewrite the distribution of the truncated standard normal distribution

$$f(x|x > a) = \frac{f(x)}{1 - \Phi(\alpha)} = \frac{(2\pi\sigma^2)^{-1/2}e^{-(x-\mu)^2/(2\sigma^2)}}{1 - \Phi(\alpha)} = \frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{1 - \Phi(\alpha)} \tag{3}$$

(Greene, Theorem 19.2) Moments of the Truncated Normal Distribution
if $x \sim N[\mu, \sigma^2]$ and a is a constant, then

$$E[x|\text{truncation}] = \mu + \sigma\lambda(\alpha), \tag{4}$$
$$Var[x|\text{truncation}] = \sigma^2[1 - \delta(\alpha)], \tag{5}$$

where $\alpha = (a - \mu)/\sigma$, $\phi(\alpha)$ is the standard normal density and

$$\lambda(\alpha) = \phi(\alpha)/[1 - \Phi(\alpha)] \text{ if truncation is } x > a, \tag{6}$$
$$\lambda(\alpha) = -\phi(\alpha)/[\Phi(\alpha)] \text{ if truncation is } x < a, \tag{7}$$

and

$$\delta(\alpha) = \lambda(\alpha)[\lambda(\alpha) - \alpha]. \tag{8}$$

An important result from the variance above is $0 < \delta(\alpha) < 1$ for all values of α. A common result which is

$$\frac{\partial\phi(\alpha)}{\partial\alpha} = -\alpha\phi(\alpha) \tag{9}$$

(6), it is also noted as the hazard function in this form (7) for the standard normal distribution.

Example[graphs of Truncated Normal Distributions]

The above was for usefulness in explaining some of the upcoming conclusions.

Censoring of the dependent variable

Censoring of the dependent variable is a common problem in microeconomic data. This occurs when a values for a certain range of the dependent variable are all transformed to a single value or reported as a single value. Some examples that have appeared in many empirical work are the following data sets. Some common examples taken from Greene.

1. Household purchases of durable goods [Tobin (1958)]
2. The number of extramarital affairs [Fair (1977, 1978)]
3. The number of hours worked by a woman in the labor force [Quester and Greene (1982)]

4. The number of arrests after release from prison [Witte (1980)]
5. Household expenditure on various commodity groups [Jarque (1987)]
6. Vacation expenditures [Melenberg and van Soest (1996)]

Question: What do you think is the common theme?

Question: Is a censored distribution continuous or discrete or a mixture (from what I just hinted at)?

Distribution Drawing (insert cartoonish drawing of a distribution)

Partially Censored Distribution,

$$y = 0 \text{ if } y^* \leq 0, \tag{10}$$
$$y = y^* \text{ if } y^* > 0, \tag{11}$$

distribution which applies to this is $y^* \sim N[\mu, \sigma^2]$ is Prob($y = 0$) =Prob($y^* \leq 0$) $= \Phi(-\mu/\sigma) = 1 - \Phi(\mu/\sigma)$, and if $y^* > 0$ then y has density y^*.

(Greene Theorem 19.3) Moments of the Censored Normal Variable
If $y \sim N[\mu, \sigma^2]$ and $y = a$ if $y^* \leq a$ or else $y = y^*$, then

$$E[y] = \Phi a + (1 - \Phi)(\mu + \sigma\lambda) \tag{12}$$

and

$$Var[y] = \sigma^2(1 - \Phi)[(1 - \delta) + (\alpha - \lambda)^2\Phi], \tag{13}$$

where

$$\Phi[(a - \mu)/\sigma] = \Phi(\alpha) = Prob(y^* \leq a) = \Phi, \lambda = \phi/(1 - \Phi), \tag{14}$$

and

$$\delta = \lambda^2 - \lambda\alpha. \tag{15}$$

The special case where $a = 0$ which is usually the case in censoring data. We then have the following results.

$$E[y|a = 0] = \Phi(\mu/\sigma)(\mu + \sigma\lambda), \tag{16}$$
$$\text{where } \lambda = \frac{\phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \tag{17}$$

If we were to censor the upper part of the distribution instead of the lower. Then we would reverse the role of Φ and $1 - \Phi$ and redefine λ. See equation (6).

and finally... we're ready to see the ...

THE TOBIT REGRESSION

The tobit regression model is

$$y_i^* = \boldsymbol{x_i'\beta} + e_i, \tag{18}$$
$$y_i = 0 \qquad \text{if } y_i^* \leq 0, \tag{19}$$
$$y_i = y_i^* \qquad \text{if } y_i^* > 0, \tag{20}$$

Expected vaule of the latent variable

$$E[y_i^*|x_i] \text{ is } \boldsymbol{x_i'\beta} \tag{21}$$

Expected value of the non-censored variable

$$E[y_i|x_i] = \Phi(\boldsymbol{x_i'\beta}/\sigma)(\boldsymbol{x_i'\beta} + \sigma\lambda_i), \tag{22}$$

where

$$\lambda_i = \frac{\phi[(0 - \boldsymbol{x_i'\beta})/\sigma]}{1 - \Phi[(0 - \boldsymbol{x_i'\beta})/\sigma]} = \frac{\phi(\boldsymbol{x_i'\beta}/\sigma)}{\Phi(\boldsymbol{x_i'\beta}\sigma)} \tag{23}$$

(Greene Theorem 19.4) Partial Effects in the Censored Regression Model
Partial effects on the latent variable(censored)

$$\frac{\partial E[y_i^*|x_i]}{\partial x_i} = \beta, \tag{24}$$

Partial effects on the whole sample

$$\frac{\partial E[y_i|x_i]}{\partial x_i} = \beta\Phi(\boldsymbol{x_i'\beta}/\sigma). \tag{25}$$

Partial effects on the uncensored sample

$$\frac{\partial E[y_i|x_i]}{\partial x_i} = \beta[\Phi_i[1 - \lambda_i(\alpha_i + \lambda_i)] + \phi_i(\alpha_i + \lambda_i)]. \tag{26}$$

Most computer programs utilize a log-likelood for censored regression.

$$lnL = \sum_{y_i>0} -\frac{1}{2}[log(2\pi) + ln\sigma^2 + \frac{(y_i - \boldsymbol{x_i'\beta})^2}{\sigma^2}] + \sum_{y_i=0} ln[1 - \Phi(\frac{\boldsymbol{x_i'\beta}}{\sigma})] \tag{27}$$

Although this is true many researchers frequently compute the OLS estimates despite their inconsistency. It is almost with certainty that OLS estimates will be smaller in absolute value then MLEs. One of the interesting trends is that MLEs can usually be approximated by dividing the OLS estimates by the proportion of the non-limit observations in the sample.

Another common occurrence is the discarding of the limit observations which is just turning the censoring problem into a truncation problem.