Simultaneous Equation Models

1 Introduction

Many economic problems involve the interaction of multiple endogenous variables within a system of equations. Estimating the parameters of such as system is typically not as simple as doing OLS equation-by-equation. Issues such as identification (whether the parameters are even estimable) and endogeneity bias are the primary topics in this chapter.

2 The Model

The simultaneous system can be written as

$$Y\Gamma + XB = E \tag{1}$$

where the variable matrices are

$$Y_{T\times M} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1M} \\ Y_{21} & Y_{22} & & Y_{2M} \\ \vdots & & \ddots & \vdots \\ Y_{T1} & Y_{T2} & \cdots & Y_{TM} \end{bmatrix};\ X_{T\times K} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1K} \\ X_{21} & X_{22} & & X_{2K} \\ \vdots & & \ddots & \vdots \\ X_{T1} & X_{T2} & \cdots & X_{TK} \end{bmatrix};\ E_{T\times M} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \cdots & \epsilon_{1M} \\ \epsilon_{21} & \epsilon_{22} & & \epsilon_{2M} \\ \vdots & & \ddots & \vdots \\ \epsilon_{T1} & \epsilon_{T2} & \cdots & \epsilon_{TM} \end{bmatrix}$$

and the coefficient matrices are

$$\Gamma_{M\times M} = \begin{bmatrix} \gamma_{11} & \gamma_{21} & \cdots & \gamma_{M1} \\ \gamma_{12} & \gamma_{22} & & \gamma_{M2} \\ \vdots & & \ddots & \vdots \\ \gamma_{1M} & \gamma_{2M} & \cdots & \gamma_{MM} \end{bmatrix};\ B_{K\times M} = \begin{bmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{M1} \\ \beta_{12} & \beta_{22} & & \beta_{M2} \\ \vdots & & \ddots & \vdots \\ \beta_{1K} & \beta_{2K} & \cdots & \beta_{MK} \end{bmatrix}.$$

Some definitions.

- $Y_{t,j}$ is the jth endogenous variable.
- $X_{t,j}$ is the jth exogenous or predetermined variable
- Equations (1) are referred to as structural equations. Γ and B are the structural parameters.

To examine the assumptions about the error terms, rewrite the E matrix as

$$\tilde{E} = vec(E) = (\epsilon_{11}, \epsilon_{21}, ..., \epsilon_{T1}, \epsilon_{12}, \epsilon_{22}, ..., \epsilon_{T2}, ..., \epsilon_{1M}, \epsilon_{2M}, ..., \epsilon_{TM})'.$$

We assume

$$\begin{aligned} E(\tilde{E}) &= 0 \\ E(\tilde{E}\tilde{E}') &= \Sigma \otimes I_T \end{aligned}$$

where the variance-covariance matrix for $\epsilon_t = (\epsilon_{t1}, \epsilon_{t2}, ..., \epsilon_{tM})'$ is

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{M1} \\ \sigma_{12} & \sigma_{22} & & \sigma_{M2} \\ \vdots & & \ddots & \vdots \\ \sigma_{1M} & \sigma_{2M} & \cdots & \sigma_{MM} \end{bmatrix}.$$

2.1 Reduced Form

The reduced-form solution to (1) is

$$Y = -XB\Gamma^{-1} + E\Gamma^{-1} = X\Pi + V$$

where $\Pi = -B\Gamma^{-1}$, $V = E\Gamma^{-1}$ and the error vector $\tilde{V} = vec(V)$ satisfies

$$\begin{aligned} E(\tilde{V}) &= 0 \\ E(\tilde{V}\tilde{V}') &= (\Gamma^{-1\prime}\Sigma\Gamma^{-1} \otimes I_T) = (\Omega \otimes I_T) \end{aligned}$$

where $\Sigma = \Gamma'\Omega\Gamma$.

2.2 Demand and Supply Example

Consider the following demand and supply equations

$$\begin{aligned} Q_t^s &= \alpha_0 + \alpha_1 P_t + \alpha_2 W_t + \alpha_3 Z_t + \epsilon_t^s \\ Q_t^d &= \beta_0 + \beta_1 P_t + \beta_3 Z_t + \epsilon_t^d \\ Q_t^s &= Q_t^d \end{aligned}$$

where Q_t^s, Q_t^d and P_t are endogenous variables and W_t and Z_t are exogenous variables. Let $Q = Q_t^s = Q_t^d$. In matrix form, the system can be written as

$$Y = \begin{bmatrix} Q_1 & P_1 \\ Q_2 & P_2 \\ \vdots & \vdots \\ Q_T & P_T \end{bmatrix};\ X = \begin{bmatrix} 1 & W_1 & Z_1 \\ 1 & W_2 & Z_2 \\ \vdots & \vdots & \vdots \\ 1 & W_T & Z_T \end{bmatrix};\ E = \begin{bmatrix} \epsilon_1^s & \epsilon_1^d \\ \epsilon_2^s & \epsilon_2^d \\ \vdots & \vdots \\ \epsilon_T^s & \epsilon_T^d \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} 1 & 1 \\ -\alpha_1 & -\beta_1 \end{bmatrix};\ B = \begin{bmatrix} -\alpha_0 & -\beta_0 \\ -\alpha_2 & 0 \\ -\alpha_3 & -\beta_3 \end{bmatrix}$$

3 Identification

Identification Question. Given data on X and Y, can we identify Γ, B and Σ?

3.1 Estimation of Π and Ω

Begin by making the standard assumptions about the reduced form $Y = X\Pi + V$:

- $\text{plim}(\frac{1}{T}X'X) = Q$
- $\text{plim}(\frac{1}{T}X'V) = 0$
- $\text{plim}(\frac{1}{T}V'V) = \Omega$.

These assumptions imply that the equation-by-equation OLS estimates of Π and Ω will be consistent.

3.2 Relationship Between (Π, Ω) and (Γ, B, Σ)

With these estimates ($\hat{\Pi}$ and $\hat{\Omega}$) in hand, the question is whether we can map back to Γ, B and Σ? We know the following

1. $\Pi = -B\Gamma^{-1}$ and
2. $\Omega = \Gamma^{-1\prime}\Sigma\Gamma^{-1}$.

To see if identification is possible, we can count the number of known elements on the left-hand side and compare with the number of unknown elements on the right-hand side.

Number of Known Elements

- KM elements in Π

- $\frac{1}{2}M(M+1)$ elements in Ω

Total $= M(K + \frac{1}{2}(M+1))$.

Number of Unknown Elements

- M^2 elements in Γ
- $\frac{1}{2}M(M+1)$ elements in Σ
- $B = KM$

Total $= M(M + K + \frac{1}{2}(M+1))$.

Therefore, we are M^2 pieces of information shy of identifying the structural parameters. In other words, there is more than one set of structural parameters that are consistent with the reduced form. We say the model is underidentified.

3.3 Identification Conditions

There are several possibilities for obtaining identification:

1. Normalization (i.e., set $\gamma_{ii} = -1$ in Γ for $i = 1, ..., m$).
2. Identities (e.g., national income accounting identity).
3. Exclusion restrictions (e.g., demand and supply shift factors).
4. Other linear (and nonlinear) restrictions (e.g., Blanchard-Quah long-run restriction).

3.3.1 Rank and Order Conditions

Begin by rewriting the i^{th} equation from $\Pi = -B\Gamma^{-1}$ in matrix form as

$$\begin{bmatrix} \Pi & I_K \end{bmatrix} \begin{bmatrix} \Gamma_i \\ B_i \end{bmatrix} = 0 \tag{2}$$

where Γ_i and B_i represent the i^{th} columns of Γ and B, respectively. Since the rank of $[\Pi \; I_K]$ equals K, (2) represents a system of K equations in $M+K-1$ unknowns (after normalization). In achieving identification, we will introduce linear restrictions as follows

$$R_i \begin{bmatrix} \Gamma_i \\ B_i \end{bmatrix} = 0 \tag{3}$$

where $Rank(R_i) = J$. Putting equations (2) and (3) together and redefining $\Delta_i = (\Gamma_i, B_i)'$ gives

$$\begin{bmatrix} (\Pi \vdots I_K) \\ R_i \end{bmatrix} \Delta_i = 0.$$

From this discussion, it is clear that R_i must provide at least $M-1$ new pieces of information. Here are the formal rank and order conditions.

1. Order Condition. The order condition states that $Rank(R_i) = J \geq M-1$ is a necessary but not sufficient condition for identification. A situation where the order condition is not sufficient is when $R_i \Delta_j = 0$. More details on the order condition below.

2. Rank Condition. The rank condition states that $Rank(R_i \Delta) = M-1$ is a necessary and sufficient condition for identification.

We can now summarize all possible identification outcomes.

- Under Identification. If either $Rank(R_i) < M-1$ or $Rank(R_i\Delta) < M-1$, the i^{th} equation is underidentified.

- Exact Identification. If $Rank(R_i) = M-1$ and $Rank(R_i\Delta) = M-1$, the i^{th} equation is exactly identified.

- Over Identification. If $Rank(R_i) > M-1$ and $Rank(R_i\Delta) = M-1$, the i^{th} equation is overidentified.

3.3.2 Identification Conditions in the Demand and Supply Example

Begin with supply and note that $M = 2$. The order condition is simple. Since all the variables are in the supply equation, there is no restriction matrix R_s so that $Rank(R_s) = 0 < 1$. The supply equation is underidentified. There is no need to look at the rank condition.

Next, consider demand. The relevant matrix equations are

$$\begin{bmatrix} (\Pi \vdots I_K) \\ R_d \end{bmatrix} \Delta_d = \begin{bmatrix} \pi_{11} & \pi_{12} & 1 & 0 & 0 \\ \pi_{21} & \pi_{22} & 0 & 1 & 0 \\ \pi_{31} & \pi_{32} & 0 & 0 & 1 \\ -- & -- & -- & -- & -- \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\beta_1 \\ -\beta_0 \\ -\beta_2 \\ -\beta_3 \end{bmatrix} = 0,$$

for which the order condition is clearly satisfied (i.e., $Rank(R_d) = 1 = M-1$). For the rank condition, we need to find the rank of

$$R_d\Delta = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\alpha_1 & -\beta_1 \\ -\alpha_0 & -\beta_0 \\ -\alpha_2 & 0 \\ -\alpha_3 & -\beta_3 \end{bmatrix} = \begin{bmatrix} -\alpha_2 & 0 \end{bmatrix}.$$

Clearly, $Rank(R_d\Delta) = 1 = M - 1$, so that the demand equation is exactly identified.

4 Limited-Information Estimation

We will consider five different limited-information estimation techniques – OLS, indirect least squares (ILS), instrumental variable (IV) estimation, two-stage least squares (2SLS) and limited-information maximum likelihood (LIML). The term limited information refers to equation-by-equation estimation, as opposed to full-information estimation which uses the linkages among the different equations.

Begin by writing the i^{th} equation as

$$\begin{aligned} Y\Gamma_i + XB_i &= \epsilon_i \\ y_i &= Y_i\gamma_i + Y_i^*\gamma_i^* + X_i\beta_i + X_i^*\beta_i^* + \epsilon_i \end{aligned}$$

where Y_i represents the vector of endogenous variables (other than y_i) in the i^{th} equation, Y_i^* represents the vector of endogenous variables excluded from the i^{th} equation, and similarly for X. Therefore, $\gamma_i^* = 0$ and $\beta_i^* = 0$ so that

$$\begin{aligned} y_i &= Y_i\gamma_i + X_i\beta_i + \epsilon_i \\ &= \begin{bmatrix} Y_i & X_i \end{bmatrix} \begin{bmatrix} \gamma_i \\ \beta_i \end{bmatrix} + \epsilon_i \\ &= Z_i\delta_i + \epsilon_i. \end{aligned}$$

4.1 Ordinary Least Squares (OLS)

The OLS estimator of δ_i is

$$\hat{\delta}_i^{OLS} = (Z_i'Z_i)^{-1}(Z_i'y_i).$$

The expected value of $\hat{\delta}_i$ is

$$E(\hat{\delta}_i^{OLS}) = \delta_i + E[(Z_i'Z_i)^{-1}Z_i'\epsilon_i].$$

However, since y_i and Y_i are jointly determined (recall Z_i contains Y_i), we cannot expect that $E(Z_i'\epsilon_i) = 0$ or $\text{plim}(Z_i'\epsilon_i) = 0$. Therefore, OLS estimates will be biased and inconsistent. This is commonly known as simultaneity or endogeneity bias.

4.2 Indirect Least Squares (ILS)

The indirect least squares estimator simply uses the consistent reduced-form estimates ($\hat{\Pi}$ and $\hat{\Omega}$) and the relations $\Pi = -B\Gamma^{-1}$ and $\Omega = \Gamma^{-1\prime}\Sigma\Gamma^{-1}$ to solve for Γ, B and Σ. The ILS estimator is only feasible if the system is exactly identified. To see this, consider the i^{th} equation as given in (2)

$$\Pi\Gamma_i = -B_i$$

where $\hat{\Pi} = (X'X)^{-1}X'Y$. Substitution gives

$$(X'X)^{-1}X' \begin{bmatrix} y_i & Y_i \end{bmatrix} \begin{bmatrix} -1 \\ \hat{\gamma}_i \end{bmatrix} = \begin{bmatrix} -\hat{\beta}_i \\ 0 \end{bmatrix}.$$

Multiplying through by $(X'X)$ gives

$$\begin{aligned} -X'y_i + X'Y_i\hat{\gamma}_i &= -X'X \begin{bmatrix} -\hat{\beta}_i \\ 0 \end{bmatrix} \Rightarrow \\ X'y_i &= X'Y_i\hat{\gamma}_i + X'X_i\hat{\beta}_i \\ &= X'Z_i\hat{\delta}_i. \end{aligned}$$

Therefore, the ILS estimator for the i^{th} equation can be written as

$$\hat{\delta}_i^{ILS} = (X'Z_i)^{-1}(X'y_i).$$

There are three cases:

1. If the i^{th} equation is exactly identified, then $X'Z_i$ is square and invertible.

2. If the i^{th} equation is underidentified, then $X'Z_i$ is not square.

3. If the i^{th} equation is overidentified, then $X'Z_i$ is not square although a subset could be used to obtain consistent albeit inefficient estimates of δ_i.

4.3 Instrumental Variable (IV) Estimation

Let W_i be an instrument matrix (dimension $T \times (K_i + M_i)$) satisfying

- $\text{plim}(\frac{1}{T}W_i'Z_i) = \Sigma_{wz}$, a finite invertible matrix

- $\text{plim}(\frac{1}{T}W_i'W_i) = \Sigma_{ww}$, a finite positive-definite matrix

- $\text{plim}(\frac{1}{T}W_i'\epsilon_i) = 0$.

Since $\text{plim}(Z_i'\epsilon_i) \neq 0$, we can instead examine

$$\begin{aligned}\text{plim}(\frac{1}{T}W_i'y_i) &= \text{plim}(\frac{1}{T}W_i'Z_i)\delta + \text{plim}(\frac{1}{T}W_i'\epsilon_i) \\ &= \text{plim}(\frac{1}{T}W_i'Z_i)\delta.\end{aligned}$$

Naturally, the instrumental variable estimator is

$$\hat{\delta}_i^{IV} = (W_i'Z_i)^{-1}(W_i'y_i)$$

which is consistent and has asymptotic variance-covariance matrix

$$asy.var.(\hat{\delta}_i^{IV}) = \frac{\sigma_{ii}}{T}[\Sigma_{wz}^{-1}\Sigma_{ww}\Sigma_{zw}^{-1}].$$

This can be estimated by

$$est.asy.var.(\hat{\delta}_i^{IV}) = \hat{\sigma}_{ii}(W_i'Z_i)^{-1}W_i'W_i(Z_i'W_i)^{-1}$$

and

$$\hat{\sigma}_{ii} = \frac{1}{T}(y_i - Z_i\hat{\delta}_i^{IV})'(y_i - Z_i\hat{\delta}_i^{IV}).$$

A degrees of freedom correction is optional. Notice that ILS is a special case of IV estimation for an exactly identified equation where $W_i = X$.

4.4 Two-Stage Least Squares (2SLS)

When an equation in the system is overidentified (i.e., $rows(X'Z_i) > cols(X'Z_i)$), a convenient and intuitive IV estimator is the two-stage least squares estimator. The 2SLS estimator works as follows:

- Stage #1. Regress Y_i on X and form $\hat{Y}_i = X\hat{\Pi}^{OLS}$.

- Stage #2. Estimate δ_i by an OLS regression of y_i on $\hat{Y}_i$ and X_i.

More formally, let $\hat{Z}_i = (\hat{Y}_i, X_i)$. The 2SLS estimator is given by

$$\hat{\delta}_i^{2SLS} = (\hat{Z}_i'\hat{Z}_i)^{-1}(\hat{Z}_i'y_i)$$

where the asymptotic variance-covariance matrix for $\hat{\delta}_i^{2SLS}$ can be estimated consistently by

$$est.asy.var(\hat{\delta}_i^{2SLS}) = \hat{\sigma}_{ii}(\hat{Z}_i'\hat{Z}_i)^{-1}$$

and

$$\hat{\sigma}_{ii} = \frac{1}{T}(y_i - Z_i\hat{\delta}_i^{2SLS})'(y_i - Z_i\hat{\delta}_i^{2SLS}).$$

4.5 Limited-Information Maximum Likelihood (LIML)

Limited-information maximum likelihood estimation refers to ML estimation of a single equation in the system. For example, if we assume normally distributed errors, then we can form the joint probability distribution function of (y_i, Y_i) and maximize it by choosing δ_i and the appropriate elements of Σ. Since the LIML estimator is more complex but asymptotically equivalent to the 2SLS estimator, it is not widely used.

Note. When the i^{th} equation is exactly identified, $\hat{\delta}_i^{ILS} = \hat{\delta}_i^{IV} = \hat{\delta}_i^{2SLS} = \hat{\delta}_i^{LIML}$.

5 Full-Information Estimation

The five estimators mentioned above are not fully efficient because they ignore cross-equation relationships between error terms and any omitted endogenous variables. We consider two fully efficient estimators below.

5.1 Three-Stage Least Squares (3SLS)

Begin by writing the system (1) as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Z_M \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_M \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_M \end{bmatrix} \Rightarrow y = \bar{Z}\delta + \epsilon$$

where $\epsilon = \tilde{E}$ and

$$E(\epsilon\epsilon') = \Sigma \otimes I_T.$$

Then, applying the principle from SUR estimation, the fully efficient estimator is

$$\hat{\delta} = (\bar{W}'(\Sigma^{-1} \otimes I)\bar{Z})^{-1}(\bar{W}'(\Sigma^{-1} \otimes I)y)$$

where $\bar{W}$ indicates an instrumental variable matrix in the form of $\bar{Z}$. Zellner and Theil (1962) suggest the following three-stage procedure for estimating δ.

- Stage #1. Calculate $\hat{Y}_i$ for each equation ($i = 1, ..., M$) using OLS and the reduced form.
- Stage #2. Use $\hat{Y}_i$ to calculate $\hat{\delta}_i^{2SLS}$ and $\hat{\sigma}_{ij} = \frac{1}{T}(y_i - Z_i\hat{\delta}_i^{2SLS})'(y_j - Z_j\hat{\delta}_j^{2SLS})$.
- Stage #3. Calculate the IV-GLS estimator

$$\begin{aligned}\hat{\delta}^{3SLS} &= [Z'(\hat{\Sigma}^{-1} \otimes X(X'X)^{-1\prime})Z]^{-1}[Z'(\hat{\Sigma}^{-1} \otimes X(X'X)^{-1\prime})y] \\ &= [\hat{Z}'(\hat{\Sigma}^{-1} \otimes I)\hat{Z}]^{-1}[\hat{Z}'(\hat{\Sigma}^{-1} \otimes I)y].\end{aligned}$$

The asymptotic variance-covariance matrix can be estimated by

$$\begin{aligned}est.asy.var(\hat{\delta}^{3SLS}) &= [Z'(\hat{\Sigma}^{-1} \otimes X(X'X)^{-1\prime})Z]^{-1} \\ &= [\hat{Z}'(\hat{\Sigma}^{-1} \otimes I)\hat{Z}]^{-1}.\end{aligned}$$

5.2 Full-Information Maximum Likelihood (FIML)

The full-information maximum likelihood estimator is asymptotically efficient. Assuming multivariate normally distributed errors, we maximize

$$\ln L(\delta, \Sigma|y, Z) = -\frac{MT}{2}\ln(2\pi) + \frac{T}{2}\ln|\Sigma|^{-1} + T\ln|\Gamma| - \frac{1}{2}(y - Z\delta)'(\Sigma^{-1} \otimes I_T)(y - Z\delta)$$

by choosing Γ, B and Σ. The FIML estimator can be computationally burdensome and has the same asymptotic distribution as the 3SLS estimator. As a result, most researchers use 3SLS.

6 Application

Consider estimating a traditional Keynesian consumption function using quarterly data between 1947 and 2003. The simultaneous system is

$$C_t = \beta_0 + \beta_1 DI_t + \epsilon_t^c \quad (4)$$

$$DI_t = \beta_2 + C_t + I_t + G_t + NX_t + \epsilon_t^y \quad (5)$$

where the variables are defined as follows:

Endogenous Variables

- C_t – Consumption.

- DI_t – Disposable Income.

Exogenous Variables

- I_t – Investment.
- G_t – Government Spending.
- NX_t – Net Exports.

The conditions for identification of (4), the more interesting equation to be estimated, are shown below. Begin by writing the system as

$$Y\Gamma + XB = Y\begin{bmatrix} 1 & -1 \\ -\beta_1 & 1 \end{bmatrix} + X\begin{bmatrix} -\beta_0 & -\beta_2 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} = E$$

where $Y_t = (C_t, DI_t)$ and $X_t = (1, I_t, G_t, NX_t)$. The order condition depends on the rank of the restriction matrix for (4),

$$R_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is obviously $Rank(R_1) = 3 > M - 1 = 1$. Therefore, the model is overidentified if the rank condition is satisfied (i.e., $Rank(R_1\Delta) = M - 1$). The relevant matrix for the rank condition is

$$R_1\Delta = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -\beta_1 & 1 \\ -\beta_0 & -\beta_2 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix},$$

which has $Rank(R_1\Delta) = 1$. Therefore, equation (4) is overidentified. Another way to see that (4) is overidentified is to solve for the reduced-form representation of C_t

$$\begin{aligned} C_t &= \beta_0 + \beta_1[\beta_2 + C_t + I_t + G_t + NX_t + \epsilon_t^y] + \epsilon_t^c \\ &= \frac{1}{1-\beta_1}[(\beta_0 + \beta_1\beta_2) + \beta_1 I_t + \beta_1 G_t + \beta_1 NX_t + (\epsilon_t^c + \beta_1\epsilon_t^y)] \\ C_t &= \pi_0 + \pi_1 I_t + \pi_1 G_t + \pi_1 NX_t + v_t. \end{aligned} \tag{6}$$

Clearly, estimation of (6) is likely to produce three distinct estimates of π_1, which reflects the overidentification problem.