Physical Capital, Human Capital, and the Health Effects of Pollution in an OLG Model

(Supplementary Document)

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Appendix A Proof of Proposition 1

Proof. Inequality (13) is the condition under which the representative agent does not invest in private education. Substituting equations (2), (4), and (15) into inequality (13) gives

$$\overline{\Phi}(z_t) < \frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\alpha)(1-\tau)}, \tag{A.1}$$

where $\overline{\Phi}(z_t) = \frac{\phi(z_t)}{1+\phi(z_t)}$ is the agent's propensity to save when she invests nothing in private education. It can be shown that $\overline{\Phi}'(z_t) < 0$ because $\phi'(z_t) < 0$ and $\overline{\Phi}(z_t) \in \left[\frac{\underline{\phi}}{1+\underline{\phi}}, \frac{\overline{\phi}}{1+\overline{\phi}}\right]$. The policy combination of τ and Δ gives rise to three cases. First, if $\frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\alpha)(1-\tau)} \leq \frac{\underline{\phi}}{1+\underline{\phi}} \iff \Delta \geq 1-(1-\alpha)\frac{\underline{\phi}}{1+\underline{\phi}}\frac{\chi\beta}{\mu}\frac{1-\tau}{\tau} \equiv f_1(\tau)$, inequality (A.1) never holds and the representative agent always invests in private education. Second, if $\frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\alpha)(1-\tau)} \geq \frac{\overline{\phi}}{1+\overline{\phi}} \iff \Delta \leq 1-(1-\alpha)\frac{\overline{\phi}}{1+\overline{\phi}}\frac{\chi\beta}{\mu}\frac{1-\tau}{\tau} \equiv f_2(\tau)$, inequality (A.1) holds for $\forall z_t \in [0,+\infty)$ and the representative agent does not invest in private education. Third, if $\frac{\underline{\phi}}{1+\underline{\phi}} < \frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\alpha)(1-\tau)} < \frac{\overline{\phi}}{1+\overline{\phi}} \iff f_2(\tau) < \Delta < f_1(\tau)$, the threshold stock of pollution $z^o(\tau,\Delta)$ satisfies

$$\overline{\Phi}(z^o) = \frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\alpha)(1-\tau)}. \tag{A.2}$$

As $\overline{\Phi}'(z_t) < 0$, for any stock of pollution $z_t \in [0, z^o]$, inequality (A.1) does not hold and the agent invests in private education; for any stock of pollution $z_t \in (z^o, +\infty)$, inequality (A.1) holds and the agent does not invest in private education.

Appendix B Proof of Proposition 2

Proof. We first check the slope of the kk locus under the PE regime and then the slope of the kk locus under the NPE regime. Under the PE regime, taking natural logs on both sides, totally differentiating equation (21a), and rearranging gives the slope of the kk locus in (z_t, k_t) space:

$$\left.\frac{dk_t}{dz_t}\right|_{PE} = \left[(1-\beta)E_{\Phi_{t+1},z_t} - \beta E_{\lambda_t,z_t}\right]\frac{k_t}{z_t}\frac{1}{1-\alpha(1-\beta)}, \tag{B.1}$$

where $E_{\Phi_{t+1},z_t} = \frac{\Phi'(z_t)}{\Phi(z_t)}z_t < 0$ is the elasticity of the propensity to save with respect to the stock of pollution when the agent invests in private education, and $E_{\lambda_t,z_t} = \frac{\lambda'(z_t)}{\lambda(z_t)}z_t < 0$ is the elasticity of the effectiveness of education expenditures with respect to the stock of pollution.

Taking natural logs on both sides of (18a) and differentiating with respect to z_t gives the expression of how the stock of pollution affects the growth rate of physical capital under the PE regime:

$$\left.\frac{\partial g_K}{\partial z_t}\right|_{PE} = \left[\frac{\Phi'(z_t)}{\Phi(z_t)}z_t\right]\frac{1}{z_t} = E_{\Phi_{t+1},z_t}\frac{1}{z_t} < 0. \tag{B.2}$$

Taking natural logs on both sides of (19a) and differentiating with respect to z_t gives the expression of how the stock of pollution affects the growth rate of human capital under the PE regime:

$$\left.\frac{\partial g_H}{\partial z_t}\right|_{PE} = \left[\beta\frac{\lambda'(z_t)}{\lambda(z_t)}z_t + \beta\frac{\Phi'(z_t)}{\Phi(z_t)}z_t\right]\frac{1}{z_t} = \left(\beta E_{\lambda_t,z_t} + \beta E_{\Phi_{t+1},z_t}\right)\frac{1}{z_t} < 0. \tag{B.3}$$

From (B.2), (B.3), and (B.1), we conclude that

$$\left.\frac{\partial g_H}{\partial z_t}\right|_{PE} \lesseqgtr \left.\frac{\partial g_K}{\partial z_t}\right|_{PE} \Longleftrightarrow \left[(1-\beta)E_{\Phi_{t+1},z_t} - \beta E_{\lambda_t,z_t}\right]\frac{1}{z_t} \gtreqless 0 \Longleftrightarrow \left.\frac{dk_t}{dz_t}\right|_{PE} \gtreqless 0,$$

which implies for any given stock of pollution under the PE regime, if the negative marginal effect of pollution on the growth rate of human capital outweighs that on the growth rate of physical capital, i.e., $\frac{\partial g_H}{\partial z_t} < \frac{\partial g_K}{\partial z_t} < 0$, the kk locus slopes up in (z_t, k_t) space; if the negative marginal effect of pollution on the growth rate of physical capital outweighs that on the growth rate of human capital, i.e., $\frac{\partial g_K}{\partial z_t} < \frac{\partial g_H}{\partial z_t} < 0$, the kk locus slopes down in (z_t, k_t) space.

We now check the slope of the kk locus under the NPE regime. Taking natural logs on both sides, totally differentiating equation (21b), and rearranging gives the slope of the kk locus under the NPE regime in (z_t, k_t) space:

$$\left.\frac{dk_t}{dz_t}\right|_{NPE} = \left[E_{\overline{\Phi}_{t+1}, z_t} - \beta E_{\lambda_t, z_t}\right] \frac{k_t}{z_t} \frac{1}{1-\alpha(1-\beta)}, \tag{B.4}$$

where $E_{\overline{\Phi}_{t+1}, z_t} = \frac{\overline{\Phi}'(z_t)}{\overline{\Phi}(z_t)} z_t < 0$ is the elasticity of the propensity to save with respect to the stock of pollution when the agent does not invest in private education.

Taking natural logs on both sides of (18b) and differentiating with respect to z_t gives

$$\left.\frac{\partial g_K}{\partial z_t}\right|_{NPE} = \left[\frac{\overline{\Phi}'(z_t)}{\overline{\Phi}(z_t)} z_t\right] \frac{1}{z_t} = E_{\overline{\Phi}_{t+1}, z_t} \frac{1}{z_t} < 0. \tag{B.5}$$

Taking natural logs on both sides of (19b) and differentiating with respect to z_t gives

$$\left.\frac{\partial g_H}{\partial z_t}\right|_{NPE} = \left[\beta \frac{\lambda'(z_t)}{\lambda(z_t)} z_t\right] \frac{1}{z_t} = \beta E_{\lambda_t, z_t} \frac{1}{z_t} < 0. \tag{B.6}$$

From (B.5), (B.6), and (B.4), we have

$$\left.\frac{\partial g_H}{\partial z_t}\right|_{NPE} \lesseqgtr \left.\frac{\partial g_K}{\partial z_t}\right|_{NPE} \Longleftrightarrow \left[E_{\overline{\Phi}_{t+1}, z_t} - \beta E_{\lambda_t, z_t}\right] \frac{1}{z_t} \gtreqless 0 \Longleftrightarrow \left.\frac{dk_t}{dz_t}\right|_{NPE} \gtreqless 0,$$

which says for any given stock of pollution under the NPE regime, if the negative marginal effect of pollution on the growth rate of human capital outweighs that on the growth rate of physical capital, i.e., $\frac{\partial g_H}{\partial z_t} < \frac{\partial g_K}{\partial z_t} < 0$, the kk locus slopes up in (z_t, k_t) space; if the negative marginal effect of pollution on the growth rate of physical capital outweighs that on the growth rate of human capital, i.e., $\frac{\partial g_K}{\partial z_t} < \frac{\partial g_H}{\partial z_t} < 0$, the kk locus slopes down in (z_t, k_t) space.

Appendix C Proof of Proposition 3

Proof. The basic idea of proving the continuity of the kk loci at the threshold stock of pollution is that at $z_t = z^o$, the values for k_t calculated from (21a) and from (21b) are the same. Because the agent's propensity to save under the *PE* regime is $\Phi(z_t) = \frac{1}{1+\chi\beta+[1/\phi(z_t)]}$ and that under the *NPE* regime is $\overline{\Phi}(z_t) = \frac{1}{1+[1/\phi(z_t)]}$, when the stock of pollution is equal to the threshold level, i.e., $z_t = z^o$, the relationship between the two propensities to save is

$$\Phi(z^o) = \frac{1}{\chi\beta + [1/\overline{\Phi}(z^o)]}. \tag{C.1}$$

As the threshold stock of pollution satisfies equation (A.2), substituting that equation into (C.1) gives

$$\Phi(z^o) = \frac{\mu}{\chi\beta} \frac{\tau(1-\Delta)}{(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)}. \tag{C.2}$$

Substituting (C.2) into (21a) to eliminate $\Phi(z^o)$ gives the ratio of physical to human capital at the threshold stock of pollution calculated from the kk locus under the *PE* regime:

$$k^o_{PE} = \left[\frac{A\mu\tau(1-\Delta)^{1-\beta}}{B\chi\beta\lambda(z^o)^{\beta}}\right]^{\frac{1}{1-\alpha(1-\beta)}}. \tag{C.3}$$

Substituting (A.2) into (21b) to eliminate $\overline{\Phi}(z^o)$ gives the ratio of physical to human capital at the threshold stock of pollution calculated from the kk locus under the *NPE* regime:

$$k^o_{NPE} = \left[\frac{A\mu\tau(1-\Delta)^{1-\beta}}{B\chi\beta\lambda(z^o)^{\beta}}\right]^{\frac{1}{1-\alpha(1-\beta)}}. \tag{C.4}$$

From (C.3) and (C.4), $k^o_{PE} = k^o_{NPE} = k^o$. Because the values for k_t corresponding to z^o, calculated either from the kk locus under the *PE* regime or from the kk locus under the *NPE* regime, are the same, we therefore conclude that there is no discontinuity on the kk loci at $z_t = z^o$.

We now prove the second part of Proposition 3. Because there is no discontinuity at the threshold stock of pollution on the kk loci under the PE and NPE regimes, from the equations for the slopes of the kk loci, (B.1) and (B.4), the difference in slopes lies in the terms associated with the propensity to save with respect to the stock of pollution. In the following proof, all of the terms associated with the stock of pollution are evaluated at z^o.

Under the PE regime, the propensity to save is $\Phi(z^o) = \frac{\phi(z^o)}{(1+\chi\beta)\phi(z^o)+1}$, so the elasticity of propensity to save with respect to the stock of pollution can be written as an expression consisting of longevity as a function of pollution and the elasticity of longevity with respect to pollution:

$$E_{\Phi,z}(z^o) = \frac{\Phi'(z^o)}{\Phi(z^o)} z^o = \frac{1}{(1+\chi\beta)\phi(z^o)+1}\left[\frac{\phi'(z^o)}{\phi(z^o)} z^o\right] = \frac{E_{\phi,z}(z^o)}{(1+\chi\beta)\phi(z^o)+1} < 0. \tag{C.5}$$

Under the NPE regime, the propensity to save is $\overline{\Phi}(z^o) = \frac{\phi(z^o)}{\phi(z^o)+1}$. The elasticity of the propensity to save with respect to the stock of pollution is

$$E_{\overline{\Phi},z}(z^o) = \frac{\overline{\Phi}'(z^o)}{\overline{\Phi}(z^o)} z^o = \frac{1}{\phi(z^o)+1}\left[\frac{\phi'(z^o)}{\phi(z^o)} z^o\right] = \frac{E_{\phi,z}(z^o)}{\phi(z^o)+1} < 0. \tag{C.6}$$

Because $0 > \frac{E_{\phi,z}(z^o)}{(1+\chi\beta)\phi(z^o)+1} > \frac{E_{\phi,z}(z^o)}{\phi(z^o)+1}$, the elasticity of the propensity to save with respect to the stock of pollution under the PE regime is larger than that under the NPE regime:

$$0 > E_{\Phi,z}(z^o) > E_{\overline{\Phi},z}(z^o). \tag{C.7}$$

Subtracting equation (B.4) from (B.1), evaluated at z^o and k^o, yields the difference in the slopes under the PE and NPE regimes at the threshold stock of pollution:

$$\left.\frac{dk_t}{dz_t}\right|_{PE} - \left.\frac{dk_t}{dz_t}\right|_{NPE} = \left[\underbrace{\left(E_{\Phi,z}(z^o) - E_{\overline{\Phi},z}(z^o)\right)}_{(+)} - \underbrace{\beta E_{\Phi,z}(z^o)}_{(-)}\right] \frac{k^o}{z^o} \frac{1}{1-\alpha(1-\beta)} > 0. \tag{C.8}$$

We thus have proved that the slope of the kk locus under the PE regime is larger than that of the kk locus under the NPE regime at the threshold stock of pollution.

Appendix D Proof of Proposition 4

Proof. When the zz locus intersects with the kk locus under the PE regime, the local dynamics are dictated by (20a) and (16). Totally differentiating (20a) and (16) around the BGP gives

$$
\begin{aligned}
dk_{t+1} &= \alpha(1-\beta)\frac{A^{1-\beta}}{B(\chi\beta)^{\beta}}\left[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)\right]^{1-\beta}\frac{\Phi(z)^{1-\beta}}{\lambda(z)^{\beta}}k^{\alpha(1-\beta)-1}dk_t \\
&\quad +\frac{A^{1-\beta}}{B(\chi\beta)^{\beta}}\left[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)\right]^{1-\beta}\frac{(1-\beta)\Phi'(z)-\beta\Phi(z)\frac{\lambda'(z)}{\lambda(z)}}{\lambda(z)^{\beta}\Phi(z)^{\beta}}k^{\alpha(1-\beta)}dz_t, \\
dz_{t+1} &= (1-\alpha)\frac{\rho}{\Delta\tau A}k^{-\alpha}dk_t+(1-\theta)dz_t,
\end{aligned}
$$

where the t subscripts are omitted to indicate that the partial derivatives are evaluated on the BGP. Substituting in (21a) and (17) evaluated on the BGP gives

$$
\begin{aligned}
dk_{t+1} &= \alpha(1-\beta)dk_t+\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]\frac{k}{z}dz_t, \\
dz_{t+1} &= (1-\alpha)\theta\frac{z}{k}dk_t+(1-\theta)dz_t.
\end{aligned}
$$

The associated Jacobian matrix is

$$
J=\begin{bmatrix} \alpha(1-\beta) & \left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]\frac{k}{z} \\ (1-\alpha)\theta\frac{z}{k} & 1-\theta \end{bmatrix}. \tag{D.1}
$$

The trace and determinant of the Jacobian matrix are

$$
TrJ = \alpha(1-\beta)+(1-\theta)<2, \tag{D.2}
$$

$$
DeJ = \alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]. \tag{D.3}
$$

Because $\alpha(1-\beta)<1$ and $1-\theta<1$, inequality (D.2) always holds, implying the summation of the eigenvalues is smaller than 2.

The sign of $(TrJ)^2-4DeJ$ determines whether the eigenvalues have imaginary parts.

From (D.2) and (D.3). We have

$$
\begin{aligned}
(TrJ)^2 - 4DeJ &= [\alpha(1-\beta)+(1-\theta)]^2 - 4\alpha(1-\beta)(1-\theta) + 4(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z} - \beta E_{\lambda,z}\right] \\
&= [\alpha(1-\beta)-(1-\theta)]^2 + 4(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z} - \beta E_{\lambda,z}\right]. \qquad \text{(D.4)}
\end{aligned}
$$

The characteristic polynomial is $p(v) = v^2 - (TrJ)v + DeJ$, and

$$
\begin{aligned}
p(1) &= 1 - TrJ + DeJ = \theta\left\{[1-\alpha(1-\beta)] - (1-\alpha)\left[(1-\beta)E_{\Phi,z} - \beta E_{\lambda,z}\right]\right\}, \\
p(-1) &= 1 + TrJ + DeJ = \theta\left\{\left(\tfrac{2}{\theta}-1\right)[1+\alpha(1-\beta)] - (1-\alpha)\left[(1-\beta)E_{\Phi,z} - \beta E_{\lambda,z}\right]\right\}.
\end{aligned}
$$

Because $\left(\frac{2}{\theta}-1\right)[1+\alpha(1-\beta)] > 1-\alpha(1-\beta)$, it is always true that $p(-1) > p(1)$.

Next, we show that the local dynamic properties of the BGP depend on the slope of the kk locus relative to that of the zz locus. From (B.1), the slope of the kk locus evaluated on the BGP is

$$
\left.\frac{dk}{dz}\right|_{PE} = \left[(1-\beta)E_{\Phi,z} - \beta E_{\lambda,z}\right]\frac{k}{z}\frac{1}{1-\alpha(1-\beta)}. \qquad \text{(D.5)}
$$

And from (17), the slope of the zz locus evaluated on the BGP is

$$
\frac{dk}{dz} = \frac{1}{1-\alpha}\frac{k}{z} > 0. \qquad \text{(D.6)}
$$

Because the kk locus may slope up or down, we break the proof into two cases. In the first case, the kk locus slopes up and $\left[(1-\beta)E_{\Phi,z} - \beta E_{\lambda,z}\right]\frac{k}{z}\frac{1}{1-\alpha(1-\beta)} > 0$. From (D.3), $DeJ = \alpha(1-\beta)(1-\theta) - (1-\alpha)\theta[(1-\beta)E_{\Phi,\lambda} - \beta E_{\lambda,z}] < \alpha(1-\beta)(1-\theta) < 1$, implying the product of the eigenvalues are smaller than 1. From (D.4), $(TrJ)^2 - 4DeJ > 0$, implying there are two real and distinct eigenvalues. As the zz locus slopes up, there are two possibilities in terms of the relative slopes of the kk locus and the zz locus. The first possibility is that the kk locus is flatter than the zz locus, so from (D.5) and (D.6) we have

$$\begin{aligned} \left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]\frac{k}{z}\frac{1}{1-\alpha(1-\beta)} &< \frac{1}{1-\alpha}\frac{k}{z}, \\ [1-\alpha(1-\beta)]-(1-\alpha)\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right] &> 0, \end{aligned}$$

so $p(1)>0$. As $p(-1)>p(1)$ always holds, $p(-1)>0$. Therefore, the two real and distinct eigenvalues lies within $(-1,1)$. The conclusion is that if the kk locus slopes up and is flatter than the zz locus on the BGP, the BGP is locally stable.

The second possibility is that the kk locus is steeper than the zz locus on the BGP. Also from (D.5) and (D.6), we have

$$\begin{aligned} \left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]\frac{k}{z}\frac{1}{1-\alpha(1-\beta)} &> \frac{1}{1-\alpha}\frac{k}{z}, \\ [1-\alpha(1-\beta)]-(1-\alpha)\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right] &< 0, \end{aligned}$$

so $p(1)<0$. Because $p(-1)>p(1)$, $p(-1)$ can be positive or negative. When $1-\alpha(1-\beta)<(1-\alpha)\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]<\left(\frac{2}{\theta}-1\right)[1+\alpha(1-\beta)]$, $p(-1)>0$. So one eigenvalue lies within $(-1,1)$, and the other is greater than 1, indicating the BGP exhibits locally saddle stability. When $1-\alpha(1-\beta)<\left(\frac{2}{\theta}-1\right)[1+\alpha(1-\beta)]<(1-\alpha)\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]$, $p(-1)<0$. So one eigenvalue is greater than 1, and the other is smaller than -1, indicating the BGP is locally unstable.

In the second case, the kk locus slopes down and by (D.5), $\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]\frac{k}{z}\frac{1}{1-\alpha(1-\beta)}<0$. Because the slope the kk locus is smaller than that of the zz locus, $p(-1)>p(1)>0$. The local dynamic property of the BGP hinges on the sign of (D.4) and on whether (D.3) is greater than 1. There are four possible combinations based on the sign of (D.4) and on whether (D.3) is greater than 1.

(1) When the eigenvalues are real and distinct, $[\alpha(1-\beta)-(1-\theta)]^2+4(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]>0$, and the product of the eigenvalues is smaller than 1, $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]<1$, the BGP is locally

stable.

(2) It can be shown that $[\alpha(1-\beta)-(1-\theta)]^2+4(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]>0$ and $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]>1$ cannot hold simultaneously. We eliminate this combination.

(3) When $[\alpha(1-\beta)-(1-\theta)]^2+4(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]<0$ and $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]<1$, the BGP features locally dampened cycles.

(4) When $[\alpha(1-\beta)-(1-\theta)]^2+4(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]<0$ and $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left[(1-\beta)E_{\Phi,z}-\beta E_{\lambda,z}\right]>1$, the BGP features locally outward cycles.

Appendix E Proof of Proposition 5

Proof. When the zz locus intersects with the kk locus under the NPE regime, the local dynamics are dictated by (20b) and (16). Totally differentiating (20b) and (16), and substituting in (21b) and (17) evaluated on the BGP gives

$$\begin{aligned} dk_{t+1} &= \alpha(1-\beta)dk_t+\left(E_{\overline{\Phi},z}-\beta E_{\lambda,z}\right)\frac{k}{z}dz_t, \\ dz_{t+1} &= (1-\alpha)\theta\frac{z}{k}dk_t+(1-\theta)dz_t, \end{aligned}$$

where the t subscripts are omitted to indicate that the partial derivatives are evaluated on the BGP. The associated Jacobian matrix is

$$J'=\begin{bmatrix} \alpha(1-\beta) & \left(E_{\overline{\Phi},z}-\beta E_{\lambda,z}\right)\frac{k}{z} \\ (1-\alpha)\theta\frac{z}{k} & 1-\theta \end{bmatrix}. \tag{E.1}$$

The trace and determinant of the Jacobian matrix are

$$TrJ' = \alpha(1-\beta)+(1-\theta)<2, \tag{E.2}$$

$$DeJ' = \alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left(E_{\overline{\Phi},z}-\beta E_{\lambda,z}\right). \tag{E.3}$$

And from (E.2) and (E.3), we have

$$\left(TrJ'\right)^2 - 4DeJ' = [\alpha(1-\beta)-(1-\theta)]^2 + 4(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right). \tag{E.4}$$

The characteristic polynomial is $p(v) = v^2 - (TrJ')v + DeJ'$, and

$$\begin{aligned} p(1) &= 1 - TrJ' + DeJ' = \theta\left\{[1-\alpha(1-\beta)] - (1-\alpha)\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right)\right\}, \\ p(-1) &= 1 + TrJ' + DeJ' = \theta\left\{\left(\tfrac{2}{\theta}-1\right)[1+\alpha(1-\beta)] - (1-\alpha)\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right)\right\}. \end{aligned}$$

Because $\left(\frac{2}{\theta}-1\right)[1+\alpha(1-\beta)] > 1-\alpha(1-\beta)$, it is always true that $p(-1) > p(1)$.

The kk locus under the NPE regime may slope up or down. The slope of the kk locus evaluated on the BGP is

$$\left.\frac{dk}{dz}\right|_{NPE} = \left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right)\frac{k}{z}\frac{1}{1-\alpha(1-\beta)}. \tag{E.5}$$

Again, we break the proof into two cases. In the first case, the kk locus under the NPE regime slopes up, so by (E.5) we have $\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right)\frac{k}{z}\frac{1}{1-\alpha(1-\beta)} > 0$. From (E.3), $DeJ' = \alpha(1-\beta)(1-\theta) - (1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) < \alpha(1-\beta)(1-\theta) < 1$, implying the product of the eigenvalues is less than 1. From (E.4), $(TrJ')^2 - 4DeJ' > 0$, implying the eigenvalues are real and distinct. If the kk locus is flatter than the zz locus on the BGP, $p(-1) > p(1) > 0$. The two real and disctinct eigenvalues lie within $(-1,1)$ and the BGP is locally stable. If the kk locus is steeper than the zz locus on the BGP, $0 > P(1)$. When $p(-1) > 0 > p(1) \iff 1-\alpha(1-\beta) < (1-\alpha)\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) < \left(\frac{2}{\theta}-1\right)[1+\alpha(1-\beta)]$, the BGP exhibits locally saddle stability; when $0 > p(-1) > p(1) \iff 1-\alpha(1-\beta) < \left(\frac{2}{\theta}-1\right)[1+\alpha(1-\beta)] < (1-\alpha)\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right)$, the BGP is locally unstable.

In the second case, the kk locus under the NPE regime slopes down, so by (E.5) we have $\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right)\frac{k}{z}\frac{1}{1-\alpha(1-\beta)} < 0$. Because the slope of the kk locus is smaller than that of the zz

locus, it is always true that $p(-1) > p(1) > 0$, but the signs of $(TrJ')^2 - 4DeJ'$ and $DeJ' - 1$ have not been determined. The following three cases may hold.

(1) When the eigenvalues are real and distinct, $[\alpha(1-\beta)-(1-\theta)]^2 + 4(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) > 0$, and the product of the eigenvalues is smaller than 1, $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) < 1$, the BGP is locally stable.

(2) When the eigenvalues are complex conjugates, $[\alpha(1-\beta)-(1-\theta)]^2 + 4(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) < 0$, and the product of the eigenvalues is less than 1, $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) < 1$, the BGP features locally dampened cycles.

(3) When the eigenvalues are complex conjugates, $[\alpha(1-\beta)-(1-\theta)]^2 + 4(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) < 0$, but their product is larger than 1, $\alpha(1-\beta)(1-\theta)-(1-\alpha)\theta\left(E_{\overline{\Phi},z} - \beta E_{\lambda,z}\right) > 1$, the BGP features locally outward cycles.

Appendix F Proof of Proposition 6

Proof. To prove that the BGP under the PE regime is preferred over the BGP under the NPE regime when multiple BGPs emerge, we show that the growth rate is higher and the stock of pollution is lower on the BGP under the PE regime than under the NPE regime. It is straightforward that the stock of pollution under the PE regime is lower because it lies to the left of the threshold stock of pollution, while the stock of pollution under the NPE regime lies to the right of the threshold stock of pollution. So next we proceed by checking and comparing the growth rates associated with BGP A and BGP C.

By equation (24), the growth rate on BGP A is

$$g_{PE}^{*,A} = \ln\left\{[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)]\frac{\rho}{\Delta\tau\theta}\frac{\Phi(z_{PE}^{*,A})}{z_{PE}^{*,A}}\right\}, \tag{F.1}$$

and by equation (25), the growth rate on BGP C is

$$g_{NPE}^{*,C} = \ln\left[(1-\tau)(1-\alpha)\frac{\rho}{\Delta\tau\theta}\frac{\overline{\Phi}(z_{NPE}^{*,C})}{z_{NPE}^{*,C}}\right]. \tag{F.2}$$

It is difficult to directly compare the growth rates on BGP A and on BGP C. So we treat the threshold stock of pollution z^o as the baseline to indirectly compare the two growth rates. Equation (C.2) gives the propensity to save under the PE regime evaluated at the threshold stock of pollution, $\Phi(z^o) = \frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)}$. Because $z_{PE}^{*,A} < z^o$ and $\Phi'(z_t) < 0$, we have $\frac{\Phi(z_{PE}^{*,A})}{z_{PE}^{*,A}} > \frac{\Phi(z^o)}{z^o} \iff g_{PE}^{*,A} > \ln\left[\frac{\rho\mu(1-\Delta)}{\chi\beta\Delta\theta}\frac{1}{z^o}\right]$ by equation (F.1). Similarly, equation (A.2) gives the propensity to save under the NPE regime evaluated at the threshold stock of pollution, $\overline{\Phi}(z^o) = \frac{\mu}{\chi\beta}\frac{\tau(1-\Delta)}{(1-\alpha)(1-\tau)}$. Because $z^o < z_{NPE}^{*,C}$ and $\overline{\Phi}'(z_t) < 0$, we have $\frac{\overline{\Phi}(z^o)}{z^o} > \frac{\overline{\Phi}(z_{NPE}^{*,C})}{z_{NPE}^{*,C}} \iff \ln\left[\frac{\rho\mu(1-\Delta)}{\chi\beta\Delta\theta}\frac{1}{z^o}\right] > g_{NPE}^{*,C}$ by equation (F.2).

Therefore, $g_{PE}^{*,A} > \ln\left[\frac{\rho\mu(1-\Delta)}{\chi\beta\Delta\theta}\frac{1}{z^o}\right] > g_{NPE}^{*,C}$. We conclude that the growth rate is higher on the BGP under the PE regime than under the NPE regime.